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A Geometric Model of Twisted Differential K-theory

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Abstract

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Byung Do Park

Advisor: Mahmoud Zeinalian

We construct a model of even twisted differential K-theory when the underlying topological twist represents a torsion class. We use smooth $U(1)$-gerbes with connection as differential twists and twisted vector bundles with connection as cycles. The model we construct satisfies the axioms of Kahle and Valentino, including functoriality, naturality of twists, and the hexagon diagram. We also construct an odd twisted Chern character of a twisted vector bundle with an automorphism. In addition to our geometric model of twisted differential K-theory, we introduce a smooth variant of the Hopkins-Singer model of differential K-theory. We prove that our model is naturally isomorphic to the Hopkins-Singer model and also to the Tradler-Wilson-Zeinalian model of differential K-theory.

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Unfortunately it is too late to thank my mother. I made a mistake in prioritizing and hence did not use the given time wisely. I genuinely hope that I do not make the same mistake again.

The reader is invited to the online preprint repository for documents [31, 32] consisting of this thesis and the most recent versions.

Contents

Chapter 1

Introduction

There has been a considerable interest in twisted and differential refinements of generalized cohomology theories. Several aspects of topology, geometry, analysis, and physics coalesce in this area, and it has provided interesting applications of ∞-categorical machinery nicely packaged by ∞-sheaves of spectra on the site of smooth manifolds. (See [6, 7] for example.)

One important question in generalized cohomology theories is whether one can represent an element of a given generalized cohomology theory of a space using geometric cocycles. For instance, an element of the singular cohomology group of a space can be represented by a singular cocycle in the space, and elements in the complex K-theory of a space can be represented by complex vector bundles over that space. However, geometric models are still unknown for most other cohomology theories such as topological modular forms, and even less is known for their twisted and differential refinements.

This work to answer this question for the case of differential refinements of twisted complex K-theory.

Twisted K-theory was first introduced by Donovan and Karoubi in [14], where twists represent torsion classes in degree 3 integral cohomology, and Rosenberg [33] for all classes. More recently twisted K-theory has received much attention because of its applications in classifying D-brane charges in string theory [38], Verlinde algebras [16], and topological insulators [18].

An archetype of differential K-theory first appeared in Karoubi [26] as the *multiplicative K-theory*. This is nowadays known as the flat subgroup of the differential K-theory. Lott [29] used Karoubi's construction to develop an index theorem, but it was mostly applications in string theory that have rekindled a considerable interest in differential K-theory (see Freed [15] for example). Perhaps one of the most remarkable steps forward in differential cohomologies is due to Hopkins and Singer [24] wherein they construct a differential extension of any exotic cohomology theory in a homotopy-theoretic way. Following this work, Bunke and Schick [8], Freed and Lott [17], Simons and Sullivan [35], and Tradler, Wilson, and Zeinalian [36, 37] all came up with more concrete and geometric models of differential K-theory.

There have been some attempts to twist differential K-theory. Carey, Mickelsson, and Wang [11] gave a construction that satisfies the square diagram and short exact sequences. Kahle and Valentino [25], in an attempt to precisely formulate the T-duality for Ramond-Ramond fields in the presence of a B-field, gave a list of axioms for twisted differential K-theory, which can be generalized as axioms for any twisted differential cohomology theory. They construct a canonical differential twist for the differential K-theory of the total space of any torus bundle ([25] Section 2.2). However, a construction of twisted differential K-theory that satisfies Kahle-Valentino axioms had not been found until very recently: In 2014, Bunke and Nikolaus [6] constructed a differential refinement of any twisted cohomology theory. Their construction of twisted differential K-theory satisfies several properties we would expect, including all Kahle-Valentino axioms, except the push-forward axiom which is not addressed in [6]. The construction of Bunke and Nikolaus provides a correct model for twisted differential cohomology theory in that their model combines twisted cohomology groups and twisted differential forms in a homotopy theoretic way, analogous to what Hopkins and Singer did in the untwisted case. However, the Bunke-Nikolaus model is as abstract as the Hopkins-Singer model. We might hope that there exists a more geometric model for a twisted differential cohomology theory, at least in the case of K-theory.

The goal of this thesis is to construct such a geometric model of twisted differential K-theory in the case that the underlying topological twists represent torsion classes in degree 3 integral cohomology. We use $U(1)$-gerbes with connection as differential twists and twisted vector bundles with connection as cycles. As is well-known, the finite dimensionality of twisted vector bundles is a sufficient condition for the twist representing a torsion class. We also have constructed a twisted differential K-theory for both torsion and non-torsion twistings using lifting bundle gerbes with connection and curving as differential twists and U_{tr}-bundle gerbe modules with connection (due to Bouwknegt, Carey, Mathai, Murray, and Stevenson [3]) as cycles, which will be discussed elsewhere. Both of our models satisfy all of the Kahle-Valentino axioms except the push-forward axiom which, together with a model of odd twisted differential K-theory, will be discussed in subsequent work.

This thesis is organized as follows. In Chapter 2, we review twisted vector bundles and set up some notation. Chapter 3 constructs the twisted Chern character form and the twisted Chern-Simons form. We also verify several properties which will be needed in later chapters. Chapter 4 defines differential twists and constructs an even twisted differential K-group. We then show that our construction is functorial, natural with respect to change of differential twist, define maps into and out of the twisted differential K-groups, and verify that our model fits into a twisted analogue of the differential K-

theory hexagon diagram à la Simons and Sullivan [35]. In Appendix A, some technical facts for constructing the "a" map in Chapter 4 are discussed. In Appendix B, we develop the odd twisted Chern character and verify that our odd twisted Chern character satisfies several properties we expect. In Appendix C, we give a technical report on aspects of classifying map models in differential K-theory. Appendix C is independent from the rest of this thesis.

Having constructed geometric models of the even twisted differential K-theory, a natural question arises: "Is there a map between our geometric model and the Bunke-Nikolaus model?" Bunke, Nikolaus and Völkl [7] answered this question for the case of untwisted differential K-theory. In this case, there is a way to obtain a sheaf of spectra on the site of smooth manifolds using the symmetric monoidal category of vector bundles with connection. In [7], they obtain a map between this sheaf of spectra into a Hopkins-Singer sheaf of spectra by the universal property of the pullback. The induced map between abelian groups is called a *cycle map*. Constructing a twisted analogue of the cycle map along this vein is our current work in progress, which we hope to complete in the near future.

Chapter 2

Review of twisted vector bundles and twisted K-theory

In this chapter, we set up notations and briefly review λ-twisted vector bundles. A good reference on twisted vector bundles is Karoubi [27], which has a broader account.

Notation 2.1. Throughout the main body of this thesis as well as Appendices A and B, all of our manifolds are connected compact smooth manifolds, and all our maps are smooth maps unless specified otherwise. In particular, X and Y always denote manifolds. We will use the notation $U_{i_1 \cdots i_n}$ to denote an n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$.

Definition 2.2. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, and λ be a $U(1)$-valued completely normalized Čech 2-cocycle. A **λ-twisted vector bundle** E of rank n over X consists of a family of product bundles $\{U_i \times \mathbb{C}^n : U_i \in$

$\mathcal{U}\}_{i\in\Lambda}$ together with transition maps

$$g_{ji} : U_{ij} \to U(n)$$

satisfying

$$g_{ii} = \mathbf{1}, \quad g_{ji} = g_{ij}^{-1}, \quad g_{kj}g_{ji} = g_{ki}\lambda_{kji}.$$

Remark 2.3. (1) Recall that a Čech cocycle $\zeta = (\zeta_{i_1\cdots i_n})$ is called *completely normalized* if $\zeta_{i_1\cdots i_n} \equiv 1$ whenever there is a repeated index, and $\zeta_{\sigma(i_1)\cdots\sigma(i_n)} = (\zeta_{i_1\cdots i_n})^{\mathrm{sign}(\sigma)}$ for any $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters.

(2) We write a λ-twisted vector bundle E of rank n as a triple

$$(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\}),$$

or a pair $(\{g_{ji}\}, \{\lambda_{kji}\})$ if the open cover $\mathcal{U}$ is clear from the context. When the rank n is zero, there exists a λ-twisted vector bundle $(\{g_{ji}^{\mathcal{O}}\}, \{\lambda_{kji}\})$ with $g_{ji}^{\mathcal{O}} = \mathbf{1}$ for all $i, j \in \Lambda$. We call it the *zero λ-twisted vector bundle.*

Definition 2.4. A **morphism** f from a λ-twisted vector bundle

$$E = (\{g_{ji}\}, \{\lambda_{kji}\})$$

of rank n to a λ-twisted vector bundle $F = (\{h_{ji}\}, \{\lambda_{kji}\})$ of rank n, with respect to the same open cover $\{U_i\}_{i\in I}$ of the base X, is a family of maps

$\{f_i : U_i \to U(n)\}_{i\in\Lambda}$ such that

$$f_j(x)g_{ji}(x) = h_{ji}(x)f_i(x) \qquad \text{for all } x \in U_{ij} \text{ and all } i, j \in \Lambda.$$

Definition 2.5. Let $E = (\{g_{ji}\}, \{\lambda_{kji}\})$ and $F = (\{h_{ji}\}, \{\lambda_{kji}\})$ be λ-twisted vector bundles of rank n and m with respect to the same covering $\mathcal{U} = \{U_i\}$ of X. The **direct sum** $E \oplus F$ is defined by $(\{g_{ji} \oplus h_{ji}\}, \{\lambda_{kji}\})$, and is a λ-twisted vector bundle of rank $n+m$. The symbol $\oplus$ between two transition maps denotes the block sum of matrices.

We denote the category of λ-twisted vector bundles over X defined on an open cover $\mathcal{U}$ by $\mathbf{Bun}(\mathcal{U}, \lambda)$. The category $\mathbf{Bun}(\mathcal{U}, \lambda)$ is an additive category with respect to the direct sum $\oplus$.

Definition 2.6. The **twisted K-theory** of X defined on an open cover $\mathcal{U}$ with a $U(1)$-gerbe twisting λ, denoted by $K^0(\mathcal{U}, \lambda)$, is the Grothendieck group of the commutative monoid $\mathrm{Vect}(\mathcal{U}, \lambda)$ of isomorphism classes of λ-twisted vector bundles on $\mathcal{U}$.

Remark 2.7. The group $K^0(\mathcal{U}, \lambda)$ for a fixed $\mathcal{U}$ depends only on the Čech cohomology class of λ. To see this, let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Recall that if an additive functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of additive categories, then it induces an isomorphism of groups $F_* : K(\mathcal{C}) \to K(\mathcal{D})$, where K denotes the K-theory functor from additive categories to abelian groups. Let σ and λ be cohomologous $U(1)$-valued Čech 2-cocycles defined on $\mathcal{U}$, i.e., $\lambda_{kji} = \sigma_{kji}\chi_{ji}\chi_{ik}\chi_{kj}$ for some Čech 1-cochain χ. There is an additive functor $\Phi : \mathbf{Bun}(\mathcal{U}, \sigma) \to \mathbf{Bun}(\mathcal{U}, \lambda)$ that is an isomorphism of categories. The functor Φ takes a σ-twisted vector bundle $(\{g_{ji}\}, \{\sigma_{kji}\})$ to a λ-twisted vector bundle $(\{g_{ji}\chi_{ji}\}, \{\lambda_{kji}\})$ and takes a morphism between σ-twisted vector bundles to itself. The inverse of Φ is defined similarly by taking a λ-twisted vector bundle $(\{g_{ji}\}, \{\lambda_{kji}\})$ to a σ-twisted vector bundle $(\{g_{ji}\chi_{ji}^{-1}\}, \{\sigma_{kji}\})$. Therefore, the induced map of groups $\Phi_* : K^0(\mathcal{U}, \sigma) \to K^0(\mathcal{U}, \lambda)$ is an isomorphism.

Definition 2.8. Let $f : Y \to X$ be a map, and $E = (\{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle defined on a covering $\mathcal{U} = \{U_i\}$ of X. Let $f^{-1}\mathcal{U}$ denote the open cover on Y consisting of open sets of the form $f^{-1}(U_i)$. The **pull-back** of the λ-twisted vector bundle E is a $(\lambda \circ f)$-twisted vector bundle $(f^{-1}\mathcal{U}, \{g_{ji} \circ f\}, \{\lambda_{kji} \circ f\})$ on Y denoted by $f^*(E)$.

Proposition 2.9. The map

$$f^* : \mathrm{Vect}(\mathcal{U}, \lambda) \to \mathrm{Vect}(f^{-1}\mathcal{U}, \lambda \circ f)$$

$$[E] \mapsto [f^* E]$$

is a monoid homomorphism with respect to $\oplus$ and therefore induces a group homomorphism

$$f^* : K^0(\mathcal{U}, \lambda) \to K^0(f^{-1}\mathcal{U}, \lambda \circ f)$$

$$[E] - [F] \mapsto [f^*E] - [f^*F].$$

Proof. The map is well-defined on $\mathrm{Vect}(\mathcal{U}, \lambda)$. Given another λ-twisted vector bundle F, we have $f^*(E) \oplus f^*(F) = f^*(E \oplus F)$. Hence f^* is a monoid homomorphism, which induces a group homomorphism f^* between K-groups. □

Chapter 3

Chern-Weil theory of twisted vector bundles

In this chapter, we review Chern-Weil theory of vector bundles and define Chern-Simons forms. We will also prove several lemmata which will be needed in subsequent chapters. Throughout this chapter let

$$\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$$

be a $U(1)$-gerbe with connection defined on an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of X. Denote by H the 3-curvature form of $\check{\lambda}$. We assume that the Dixmier-Douady class of λ is a torsion class in $H^3(X;\mathbb{Z})$. We refer the reader to Appendix A for the language of $U(1)$-gerbes with connection used in this thesis as well as Gawędzki and Reis [19] for more details.

Definition 3.1. Let $\check{\lambda}$ be as above, and let $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle of rank n. A **connection** on E compatible with $\check{\lambda}$ is a family $\Gamma = \{\Gamma_i \in \Omega^1(U_i; \mathfrak{u}(n))\}_{i\in\Lambda}$ satisfying that

$$\Gamma_i - g_{ji}^{-1}\Gamma_j g_{ji} - g_{ji}^{-1} dg_{ji} = -A_{ji} \cdot \mathbf{1}. \tag{3.1}$$

Here $\mathfrak{u}(n)$ denotes the Lie algebra of $U(n)$, and $\mathbf{1}$ the identity matrix.

Lemma 3.2. In the notation of Definition 3.1, $A_{ji} \cdot \mathbf{1} - A_{ki} \cdot \mathbf{1} + A_{kj} \cdot \mathbf{1} = \lambda_{kji}^{-1} d\lambda_{kji} \cdot \mathbf{1}$.

Proof.

$$\begin{aligned}
(A_{ji} - A_{ki} + A_{kj}) \cdot \mathbf{1} &= -(\Gamma_i - g_{ji}^{-1}\Gamma_j g_{ji} - g_{ji}^{-1} dg_{ji}) \\
&\quad + (\Gamma_i - g_{ki}^{-1}\Gamma_k g_{ki} - g_{ki}^{-1} dg_{ki}) + A_{kj} \cdot \mathbf{1} \\
&= g_{ji}^{-1}(g_{kj}^{-1}\Gamma_k g_{kj} + g_{kj}^{-1} dg_{kj} - A_{kj} \cdot \mathbf{1}) g_{ji} + g_{ji}^{-1} dg_{ji} \\
&\quad - \lambda_{kji} g_{ij} g_{jk} \Gamma_k g_{kj} g_{ji} \lambda_{kji}^{-1} - g_{ki}^{-1} dg_{ki} + A_{kj} \cdot \mathbf{1} \\
&= g_{ji}^{-1} g_{kj}^{-1} \Gamma_k g_{kj} g_{ji} + g_{ji}^{-1} g_{kj}^{-1} dg_{kj} g_{ji} - A_{kj} \cdot \mathbf{1} + g_{ji}^{-1} dg_{ji} \\
&\quad - g_{ij} g_{jk} \Gamma_k g_{kj} g_{ji} - g_{ki}^{-1} dg_{ki} + A_{kj} \cdot \mathbf{1} \\
&= g_{ji}^{-1} g_{kj}^{-1} (-g_{kj} dg_{ji} + dg_{ki} \lambda_{kji} + g_{ki} d\lambda_{kji}) \\
&\quad + g_{ji}^{-1} dg_{ji} - g_{ki}^{-1} dg_{ki} \\
&= g_{ik} g_{kj} g_{ji} g_{ji}^{-1} g_{kj}^{-1} dg_{ki} + g_{ji}^{-1} g_{kj}^{-1} g_{ki} d\lambda_{kji} - g_{ki}^{-1} dg_{ki} \\
&= \lambda_{kji}^{-1} d\lambda_{kji} \cdot \mathbf{1}.
\end{aligned}$$

□

Remark 3.3. Let $\check{\lambda}$ be as above. For any λ-twisted vector bundle E, there exists a connection on E associated with $\check{\lambda}$. See [27], p.244.

Definition 3.4. Let $\check{\lambda}$ be as above, and (E, Γ) be a λ-twisted vector bundle $(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ of rank n with a connection Γ compatible with $\check{\lambda}$. The **curvature form** of Γ is the family $R = \{R_i \in \Omega^2(U_i; \mathfrak{u}(n))\}_{i\in\Lambda}$, where $R_i := d\Gamma_i + \Gamma_i \wedge \Gamma_i$.

Lemma 3.5. For each $m \in \mathbb{Z}^+$, the differential forms $\operatorname{tr}[(R_i - B_i \cdot \mathbf{1})^m]$ over the open sets U_i glue together to define a global differential form on X.

Proof. From (3.1), it follows that $R_i = g_{ji}^{-1} R_j g_{ji} - dA_{ji} \cdot \mathbf{1}$. Then

$$\begin{aligned}
\operatorname{tr}[(R_i - B_i \cdot \mathbf{1})^m] &= \operatorname{tr}\left[(g_{ji}^{-1} R_j g_{ji} - dA_{ji} \cdot \mathbf{1} - B_i \cdot \mathbf{1})^m\right] \\
&= \operatorname{tr}\left[(g_{ji}^{-1} R_j g_{ji} - B_j \cdot \mathbf{1})^m\right] \\
&= \operatorname{tr}\left[\sum_{r=0}^{m} {}_mC_r g_{ji}^{-1} R_j^{m-r} g_{ji} (-1)^r (B_j \cdot \mathbf{1})^r\right] \\
&\overset{*}{=} \operatorname{tr}\left[\sum_{r=0}^{m} {}_mC_r R_j^{m-r} (-1)^r (B_j \cdot \mathbf{1})^r\right] = \operatorname{tr}[(R_j - B_j \cdot \mathbf{1})^m],
\end{aligned}$$

where ${}_mC_r$ is the binomial coefficient m choose r, and at $*$, we have used $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and the fact that $B_j \cdot \mathbf{1}$ commutes with other matrices. □

Definition 3.6. Let $\check{\lambda}$ be as above, H be the 3-curvature of $\check{\lambda}$, and (E, Γ) be a λ-twisted vector bundle with connection compatible with $\check{\lambda}$. For $m > 0$, the m^{th} **twisted Chern character form** is defined by

$$\operatorname{ch}_{(m)}(\Gamma)(x) := \operatorname{tr}(R_i(x) - B_i(x) \cdot \mathbf{1})^m \qquad x \in U_i.$$

When $m = 0$, define $\mathrm{ch}_{(0)}(\Gamma)$ to be the rank of E. The **total twisted Chern character form** is defined by

$$\mathrm{ch}(\Gamma) := \mathrm{rank}(E) + \sum_{m=1}^{\infty} \frac{1}{m!}\mathrm{ch}_{(m)}(\Gamma),$$

which will be sometimes denoted by $\mathrm{ch}(E, \Gamma)$.

Proposition 3.7. The total twisted Chern character form $\mathrm{ch}(\Gamma)$ is $(d+H)$-closed.

Proof. We split the calculation into several parts. First,

$$\begin{aligned}
d\mathrm{tr}(R_i^p) &= \mathrm{tr}(dR_i \wedge \underbrace{R_i \wedge \cdots \wedge R_i}_{p-1} + \cdots + \underbrace{R_i \wedge \cdots \wedge R_i}_{p-1} \wedge dR_i) \\
&= \mathrm{tr}((R_i \wedge \Gamma_i - \Gamma_i \wedge R_i) \wedge \underbrace{R_i \wedge \cdots \wedge R_i}_{p-1} + \cdots \\
&\qquad + \underbrace{R_i \wedge \cdots \wedge R_i}_{p-1} \wedge (R_i \wedge \Gamma_i - \Gamma_i \wedge R_i)) \\
&= \mathrm{tr}(\underbrace{R_i \wedge \cdots \wedge R_i}_{p} \wedge \Gamma_i - \Gamma_i \wedge \underbrace{R_i \wedge \cdots \wedge R_i}_{p}) \\
&= \sum_{k,l} \big((R_i^p)_{kl}(\Gamma_i)_{lk} - (\Gamma_i)_{kl}(R_i^p)_{lk}\big) = 0.
\end{aligned}$$

Now we verify that $d\mathrm{ch}_{(m)}(\Gamma) = m\mathrm{ch}_{(m-1)}(\Gamma)H$ for all $m \geq 1$:

$$\begin{aligned}
d\mathrm{ch}_{(m)}(\Gamma) &= d\mathrm{tr}(R_i - B_i \cdot \mathbf{1})^m \\
&= d\mathrm{tr}\left(\sum_{r=0}^{m} {}_mC_r R_i^{m-r}(-1)^r(B_i \cdot \mathbf{1})^r\right) \\
&= \sum_{r=0}^{m} {}_mC_r(-1)^r d\mathrm{tr}\Big(R_i^{m-r}(B_i \cdot \mathbf{1})^r\Big) \\
&= \sum_{r=0}^{m} {}_mC_r(-1)^r \mathrm{tr}\Big(d(R_i^{m-r})(B_i \cdot \mathbf{1})^r \\
&\qquad + R_i^{m-r} \cdot r \cdot (dB_i \cdot \mathbf{1}) \wedge (B_i \cdot \mathbf{1})^{r-1}\Big) \\
&= 0 + \sum_{r=0}^{m} {}_mC_r(-1)^r \cdot r \cdot \mathrm{tr}\Big(R_i^{m-r}(dB_i \cdot \mathbf{1}) \wedge (B_i \cdot \mathbf{1})^{r-1}\Big) \\
&= \sum_{r=1}^{m} \frac{m \cdot (m-1)!}{(m-1-(r-1))!(r-1)! \cdot r}(-1)^r \cdot r \\
&\qquad \cdot \mathrm{tr}\Big(R_i^{m-r}(B_i \cdot \mathbf{1})^{r-1}\Big) \wedge H \\
&= -m\sum_{r=1}^{m} {}_{m-1}C_{r-1}\mathrm{tr}\Big(R_i^{m-r}(-1)^{r-1}(B_i \cdot \mathbf{1})^{r-1}\Big) \wedge H \\
&= -m \cdot \mathrm{ch}_{(m-1)}(\Gamma) \wedge H.
\end{aligned}$$

Hence

$$\begin{aligned}
d\mathrm{ch}(\Gamma) &= 0 - nH - \frac{1}{2!} \cdot 2 \cdot \mathrm{ch}_{(1)}(\Gamma)H - \cdots - \frac{1}{m!} \cdot m \cdot \mathrm{ch}_{(m-1)}(\Gamma)H - \cdots \\
&= -\mathrm{ch}(\Gamma)H.
\end{aligned}$$

□

Proposition 3.8. The m^{th} twisted chern character form is additive for all $m \geq 0$, i.e.,

$$\mathrm{ch}_{(m)}(\Gamma^E \oplus \Gamma^F) = \mathrm{ch}_{(m)}(\Gamma^E) + \mathrm{ch}_{(m)}(\Gamma^F).$$

Proof.

$$\begin{aligned}
\mathrm{ch}_{(m)}(\Gamma^E \oplus \Gamma^F) &= \mathrm{tr}\left(\begin{pmatrix} R_i^E & O \\ O & R_i^F \end{pmatrix} - B_i \cdot \mathbf{1}\right)^m, \\
&\quad \text{where } O \text{ is the zero matrix,} \\
&= \mathrm{tr}\begin{pmatrix} (R_i^E - B_i \cdot \mathbf{1})^m & O \\ O & (R_i^F - B_i \cdot \mathbf{1})^m \end{pmatrix} \\
&= \mathrm{ch}_{(m)}(\Gamma^E) + \mathrm{ch}_{(m)}(\Gamma^F).
\end{aligned}$$

□

Definition 3.9. Let $\check{\lambda}$ be as above, and (E, Γ) be a λ-twisted vector bundle $(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ of rank n with a connection Γ compatible with $\check{\lambda}$. Let $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ be any map provided that $\mathcal{V} = f^{-1}\mathcal{U}$. The **pullback** of (E, Γ) along f is $f^*(E)$ together with the family

$$f^*\Gamma := \{f^*\Gamma_i\}_{i \in \Lambda},$$

where $f^*\Gamma_i \in \Omega^1(f^{-1}(U_i); \mathfrak{u}(n))$ is defined by entrywise pullback.

Proposition 3.10. $f^*\Gamma$ is a connection on the $(\lambda \circ f)$-twisted vector bundle $(\mathcal{V}, \{g_{ji} \circ f\}, \{\lambda_{kji} \circ f\})$ of rank n compatible with

$$f^*\check{\lambda} = (\{\lambda_{kji} \circ f\}, \{f^*A_{ji}\}, \{f^*B_i\}).$$

Proof. We need to check (3.1) holds for $f^*\Gamma$:

$$f^*\Gamma_i(y) - (g_{ji} \circ f(y))^{-1} f^*\Gamma_j(y)(g_{ji} \circ f)(y) + (g_{ji} \circ f(y))^{-1} d(g_{ji} \circ f)(y)$$

$$= \Gamma_i(f(y))(f_*) - (g_{ji}(f(y)))^{-1}\Gamma_j(f(y))(f_*)g_{ji}(f(y))$$

$$+ (g_{ji}(f(y)))^{-1} d(g_{ji}(f(y)))(f_*)$$

$$= -A_{ji}(f(y))(f_*) \cdot \mathbf{1} \quad \text{because } \Gamma \text{ is a connection.}$$

$$= -f^*A_{ji}(y) \cdot \mathbf{1}.$$

Here f_* denotes the pushforward on the tangent space. □

Proposition 3.11. Let $\check{\lambda}$ be as above, $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle of rank n, and Γ be a connection on E compatible with $\check{\lambda}$. For any map $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ provided that $\mathcal{V} = f^{-1}\mathcal{U}$,

$$f^*\mathrm{ch}(\Gamma) = \mathrm{ch}(f^*\Gamma).$$

Proof.

$$f^*\mathrm{ch}(\Gamma) = n + \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{tr}\left[(f^*(d\Gamma_i + \Gamma_i \wedge \Gamma_i) - f^*B_i \cdot \mathbf{1})^m\right] = \mathrm{ch}(f^*\Gamma).$$

□

Proposition 3.12. Let $\check{\lambda}$ be as above and let $\varphi : E \to F$ be an isomorphism of λ-twisted vector bundles over X with respect to the same open cover $\mathcal{U}$. Let Γ^E and Γ^F be connections on E and F, respectively, both are associated with $\check{\lambda}$. Then $\mathrm{ch}(\Gamma^F) = \mathrm{ch}(\varphi^*\Gamma^F)$.

Proof. The connection $\varphi^*\Gamma^F$ is by definition the family $\{(\varphi^*\Gamma^F)_i\}_{i\in\Lambda}$, where

$$(\varphi^*\Gamma^F)_i := \varphi_i^{-1}\Gamma_i^F\varphi_i + \varphi_i^{-1}d\varphi_i.$$

The curvature form becomes:

$$\begin{aligned}
(\varphi^*R^F)_i &= d(\varphi^*\Gamma^F)_i + (\varphi^*\Gamma^F)_i \wedge (\varphi^*\Gamma^F)_i \\
&= d(\varphi_i^{-1}\Gamma_i^F\varphi_i \\
&\qquad + \varphi_i^{-1}d\varphi_i) + (\varphi_i^{-1}\Gamma_i^F\varphi_i + \varphi_i^{-1}d\varphi_i) \wedge (\varphi_i^{-1}\Gamma_i^F\varphi_i + \varphi_i^{-1}d\varphi_i) \\
&= d\varphi_i^{-1}\Gamma_i^F\varphi_i + \varphi_i^{-1}d\Gamma_i^F\varphi_i - \varphi_i^{-1}\Gamma_i^F d\varphi_i + d(\varphi_i^{-1})d\varphi_i \\
&\qquad + \varphi_i^{-1}\Gamma_i^F \wedge \Gamma_i^F\varphi_i + (\varphi_i^{-1}\Gamma_i^F\varphi_i)\varphi_i^{-1}d\varphi_i + \varphi_i^{-1}d\varphi_i(\varphi_i^{-1}\Gamma_i^F\varphi_i) \\
&\qquad + \varphi_i^{-1}d\varphi_i\varphi_i^{-1}d\varphi_i \\
&= \varphi_i^{-1}(d\Gamma_i^F + \Gamma_i^F \wedge \Gamma_i^F)\varphi_i = \varphi_i^{-1}R_i^F\varphi_i, \quad \text{since } \varphi_i^{-1}d\varphi_i = -d\varphi_i^{-1}\varphi_i.
\end{aligned}$$

Hence $\mathrm{tr}((\varphi^*R^F)_i - B_i\cdot\mathbf{1})^m = \mathrm{tr}(R_i^F - B_i\cdot\mathbf{1})^m$, and accordingly, $\mathrm{ch}(\varphi^*\Gamma^F) = \mathrm{ch}(\Gamma^F)$. □

Remark 3.13. We shall prove in Proposition 3.24 that the total *twisted Chern character* in twisted de Rham cohomology group is independent of the choice of connection. By the above Proposition, the total twisted Chern character E and F are the same, if E and F are isomorphic λ-twisted vector bundles.

Proposition 3.14. Let

$$\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \quad \text{and} \check{\lambda}' = (\{\lambda'_{kji}\}, \{A'_{ji}\}, \{B'_i\})$$

be two $U(1)$-gerbes with connection defined on an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ over X. Suppose $\check{\lambda}$ and $\check{\lambda}'$ are cohomologous as Deligne 2-cocycles such that $\check{\lambda}' = \check{\lambda} + D\check{\alpha}$, where $\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\}) \in \check{C}^1(\mathcal{U}, \Omega^1)$. (See Remark A.2 for the definition of D.) Let $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle of rank n and $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ a connection on E compatible with $\check{\lambda}$. Define a λ'-twisted vector bundle E' with connection Γ' compatible with $\check{\lambda}'$ by

$$E' := (\mathcal{U}, \chi_{ji} g_{ji}, \lambda'_{kji})$$

$$\Gamma' := \{\Gamma'_i\}_{i\in\Lambda}, \text{ where } \Gamma'_i := \Gamma_i + \Pi_i \cdot \mathbf{1}.$$

Then $\mathrm{ch}(\Gamma) = \mathrm{ch}(\Gamma')$.

Remark 3.15. Since $\check{\lambda}$ and $\check{\lambda}'$ are cohomologous, their 3-curvatures are the same (see Appendix A).

Proof of Proposition 3.14. From

$$\begin{aligned} R'_i = d\Gamma'_i + \Gamma'_i \wedge \Gamma'_i &= d(\Gamma_i + \Pi_i \cdot \mathbf{1}) + (\Gamma_i + \Pi_i \cdot \mathbf{1}) \wedge (\Gamma_i + \Pi_i \cdot \mathbf{1}) \\ &= d\Gamma_i + d\Pi_i \cdot \mathbf{1} + \Gamma_i \wedge \Gamma_i + \Gamma_i \wedge \Pi_i \cdot \mathbf{1} \\ &\quad + \Pi_i \cdot \mathbf{1} \wedge \Gamma_i + \Pi_i \cdot \mathbf{1} \wedge \Pi_i \cdot \mathbf{1} \\ &= R_i + d\Pi_i \cdot \mathbf{1}, \end{aligned}$$

it follows that

$$\mathrm{ch}_{(m)}(E,\Gamma) - \mathrm{ch}_{(m)}(E',\Gamma') = \mathrm{tr}(R_i - B_i \cdot \mathbf{1})^m - \mathrm{tr}(R'_i - B'_i \cdot \mathbf{1})^m = 0$$

since $B'_i - B_i = d\Pi_i$. □

Notation 3.16. Given $\check{\lambda}$ and $\xi \in \Omega^2(X; i\mathbb{R})$, we denote by $\check{\lambda}_\xi$ the $U(1)$-gerbe with connection $(\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i + \xi|_{U_i}\})$. Let E be a λ-twisted vector bundle and $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ be a connection on E associated with $\check{\lambda}$. We denote the same family $\{\Gamma_i\}_{i\in\Lambda}$ on E that is associated with $\check{\lambda}_\xi$ as a connection on E by Γ_ξ. We also denote $\xi|_{U_i}$ by ξ_i.

Proposition 3.17. Let $\check{\lambda}$ be as above, E be a λ-twisted vector bundle, Γ be a connection on E, and $\xi \in \Omega^2(X; i\mathbb{R})$. Then $\mathrm{ch}(\Gamma_{-\xi}) = \mathrm{ch}(\Gamma) \wedge \exp(\xi)$.

Proof.

$$\begin{aligned}
\mathrm{ch}(\Gamma_{-\xi}) &= \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{ch}_{(m)}(\Gamma_{-\xi}) = \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{tr}\left[(R_i - B_i \cdot \mathbf{1} + \xi_i \cdot \mathbf{1})^m\right] \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{tr}\left(\sum_{r=0}^{m} \ {}_mC_r (R_i - B_i \cdot \mathbf{1})^{m-r} \xi_i^r \cdot \mathbf{1}\right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\sum_{r=0}^{m} \left({}_mC_r \mathrm{tr}(R_i - B_i \cdot \mathbf{1})^{m-r}\right) \wedge \xi_i^r \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\sum_{r=0}^{m} \left(\frac{m!}{(m-r)!r!}\mathrm{ch}_{(m-r)}(\Gamma)\right) \wedge \xi_i^r \\
&= \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{\mathrm{ch}_{(m-r)}(\Gamma)}{(m-r)!} \wedge \frac{\xi_i^r}{r!} \\
&= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{\mathrm{ch}_{(m)}(\Gamma)}{(m)!} \wedge \frac{\xi_i^r}{r!} \quad \text{since } \sum_{m=0}^{\infty}\sum_{r=0}^{m} a_{r,m-r} = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} a_{r,m} \\
&= \mathrm{ch}(\Gamma) \wedge \exp(\xi).
\end{aligned}$$

□

Now we discuss the Chern-Simons transgression form in the twisted case.

Lemma 3.18. Let $\check{\lambda}$ be as above, and let Γ_0 and Γ_1 be connections on a λ-twisted vector bundle $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ such that both are compatible with $\check{\lambda}$. Then for each $t \in \mathbb{R}$,

$$\Gamma_t := (1-t)\Gamma_0 + t\Gamma_1$$

is a connection on E compatible with $\check{\lambda}$, i.e., the space of $\check{\lambda}$-compatible connections on E is an affine space modelled over $\Omega^1(X; \mathrm{End}(E))$.

Proof.

$$
\begin{aligned}
\Gamma_{ti} - g_{ji}^{-1}\Gamma_{tj}g_{ji} - g_{ji}^{-1}dg_{ji} &= (1-t)\Gamma_{0i} + t\Gamma_{1i} \\
&\quad - g_{ji}^{-1}\big((1-t)\Gamma_{0j} + t\Gamma_{1j}\big)g_{ji} - g_{ji}^{-1}dg_{ji} \\
&= (1-t)\Gamma_{0i} - (1-t)g_{ji}^{-1}\Gamma_{0j}g_{ji} - (1-t)g_{ji}^{-1}dg_{ji} \\
&\quad + t\Gamma_{1i} - tg_{ji}^{-1}\Gamma_{1j}g_{ji} - tg_{ji}^{-1}dg_{ji} \\
&= -(1-t)A_{ji}\cdot\mathbf{1} - tA_{ji}\cdot\mathbf{1} = -A_{ji}\cdot\mathbf{1}.
\end{aligned}
$$

If Γ and Γ' are two different $\check{\lambda}$-compatible connections on E, we have $\Gamma_i - \Gamma'_i = g_{ji}^{-1}(\Gamma_j - \Gamma'_j)g_{ji}$, so the space of $\check{\lambda}$-compatible connections on E is an affine space modelled over $\Omega^1(X; \mathrm{End}(E))$. Notice that $\mathrm{End}(E)$ is an ordinary vector bundle. □

Corollary 3.19. Let Γ_0 and Γ_1 be as in Lemma 3.18. Let α_t and γ_t with $t \in I$ be two paths of connections each starting at Γ_0 and ending at Γ_1 and both α_t and γ_t are compatible with $\check{\lambda}$ for all $t \in I$. Then there exists a bigon of connections with edges α_t and γ_t such that every point on the bigon is a $\check{\lambda}$-compatible connection on E.

Proof. By Lemma 3.18, for each fixed $t \in I$ and $s \in I$, the connection $(1-s)\alpha_t + s\gamma_t$ is $\check{\lambda}$-compatible. □

Notation 3.20. We shall denote the projection map $X \times I \to X$ onto the first factor by p.

Lemma 3.21. Let $\check{\lambda}$ be as above, E be a λ-twisted vector bundle

$$(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\}),$$

and Γ_t be a connection on E compatible with $\check{\lambda}$ for each $t \in I$. Then the family $\{\widetilde{\Gamma}_i\}_{i\in\Lambda}$ defined by $\widetilde{\Gamma}_i(x,t) := (p^*\Gamma_t)(x,t)$ is a connection on a $(\lambda \circ p)$-twisted vector bundle $p^*E = (\mathcal{U} \times I, \{g_{ji} \circ p\}, \{\lambda_{kji} \circ p\})$ compatible with the pull-back $U(1)$-gerbe with connection $p^*\check{\lambda} = (\{\lambda_{kji} \circ p\}, \{p^*A_{ji}\}, \{p^*B_i\})$.

Proof.

$$\begin{aligned}
&\widetilde{\Gamma}_i(x,t) - \big(g_{ji} \circ p(x,t)\big)^{-1}\widetilde{\Gamma}_j(x,t)\big(g_{ji} \circ p(x,t)\big) \\
&\qquad - \big(g_{ji} \circ p(x,t)\big)^{-1} d\big(g_{ji} \circ p\big)(x,t) \\
&= \big(\Gamma_t(x) - (g_{ji}(x))^{-1}\Gamma_j(x)g_{ji}(x) - (g_{ji}(x))^{-1}dg_{ji}(x)\big)(p_*)_{(x,t)} \\
&= -(A_{ji}(x)(p_*)_{(x,t)}) \cdot \mathbf{1} = -p^*A_{ji}(x,t) \cdot \mathbf{1},
\end{aligned}$$

□

We refer the reader to Bott and Tu [2] for an account of the integration along the fiber.

Definition 3.22. Let $\check{\lambda}$ be as above, E be a λ-twisted vector bundle over X, and $\gamma : t \mapsto \Gamma_t$ be a path of connections on E such that each Γ_t is compatible with $\check{\lambda}$. The **twisted Chern-Simons form** of γ is the integration along the fiber:

$$\mathrm{cs}(\gamma) := \int_I \mathrm{ch}(\widetilde{\Gamma}) \in \Omega^{\mathrm{odd}}(X;\mathbb{C}),$$

where $\widetilde{\Gamma}$ is the connection on p^*E defined by $\widetilde{\Gamma}(x,t) = (p^*\Gamma_t)(x,t)$.

The following lemma is certainly well-known, but we did not find a reference.

Lemma 3.23. Let E be a smooth fiber bundle over X with fiber F a compact oriented smooth k-manifold with corners. Let $\int_F : \Omega^\bullet(E;\mathbb{C}) \to \Omega^{\bullet-k}(X;\mathbb{C})$ be the integration along the fiber map and $\omega \in \Omega^n(E;\mathbb{C})$ for $n \geq k$. Then

$$d\int_F \omega = \int_F d\omega + (-1)^{n-k}\int_{\partial F}\omega. \tag{3.2}$$

Proof. Recall that the map $\int : \Omega^n(X;\mathbb{C}) \to C^n(X,\mathbb{C})$ is injective. Therefore, it suffices to show that both sides of (3.2) are the same as cochains, so it is enough to verify that the integral of each side over a $(n-k+1)$-simplex yields the same value. Since $\partial(\triangle^{n-k+1} \times F) = \partial\triangle^{n-k+1} \times F + (-1)^{n-k+1}\triangle^{n-k+1} \times$

∂F, it follows that

$$\begin{aligned}\int_{\triangle^{n-k+1}} d\int_F \omega &= \int_{\partial\triangle^{n-k+1}}\int_F \omega = \int_{\partial\triangle^{n-k+1}\times F}\omega \\ &= \int_{\partial(\triangle^{n-k+1}\times F)-(-1)^{n-k+1}\triangle^{n-k+1}\times\partial F}\omega \\ &= \int_{\partial(\triangle^{n-k+1}\times F)}\omega - (-1)^{n-k+1}\int_{\triangle^{n-k+1}\times\partial F}\omega \\ &= \int_{\triangle^{n-k+1}\times F} d\omega + (-1)^{n-k}\int_{\triangle^{n-k+1}}\int_{\partial F}\omega.\end{aligned}$$

Hence the result. □

Proposition 3.24. Let $\check{\lambda}$ be as above, $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle of rank n, and $\gamma : t \mapsto \Gamma_t$ be a path of connections on E joining Γ_0 and Γ_1 such that each Γ_t is compatible with $\check{\lambda}$. Then

$$\mathrm{ch}(\Gamma_0) - \mathrm{ch}(\Gamma_1) = (d+H)\mathrm{cs}(\gamma).$$

Proof. Let $\widetilde{\Gamma}$ be a connection on p^*E defined by $\widetilde{\Gamma}(x,t) = (p^*\Gamma_t)(x,t)$. By Lemma 3.23,

$$\begin{aligned}d\int_I \mathrm{ch}(\widetilde{\Gamma}) &= \int_I d\mathrm{ch}(\widetilde{\Gamma}) - \int_{\partial I}\mathrm{ch}(\widetilde{\Gamma}) \\ &= -\int_I \mathrm{ch}(\widetilde{\Gamma})\wedge p^*H - \int_{\partial I}\mathrm{ch}(\widetilde{\Gamma}) \\ &= -H\wedge\big(\int_I \mathrm{ch}(\widetilde{\Gamma})\big) + \mathrm{ch}(\Gamma_0) - \mathrm{ch}(\Gamma_1).\end{aligned}$$

Hence the result. □

Remark 3.25. We refer the reader to Atiyah and Segal [1] for more details on twisted cohomology. Recall that the $\mathbb{Z}_2$-graded sequence of differential forms $\cdots \to \Omega^{\text{even}}(X) \xrightarrow{d+H} \Omega^{\text{odd}}(X) \xrightarrow{d+H} \cdots$ is a complex if H is a closed 3-form on X. The *twisted de Rham cohomology* of X with the choice of closed 3-form H is the cohomology of this complex, and denote it by $H_H^{\text{even/odd}}(X)$. If closed 3-forms H and H' are cohomologous, i.e. $H' = H + d\xi$, the multiplication by $\exp(\xi)$ induces an isomorphism $H_H^\bullet(X) \to H_{H'}^\bullet(X)$.

Definition 3.26. The **twisted total Chern character** of E, denoted by $\text{ch}(E)$, is the twisted cohomology class of $\text{ch}(\Gamma)$ for any connection Γ on E.

Proposition 3.27. The assignment

$$\text{ch} : K^0(\mathcal{U}, \lambda) \to H_H^{\text{even}}(X; \mathbb{C})$$

$$[E] - [F] \mapsto [\text{ch}(\Gamma^E)] - [\text{ch}(\Gamma^F)],$$

with $(\{A_{ji}\}, \{B_i\})$ a representative connection on λ and Γ^E and Γ^F representative connections on λ-twisted vector bundles E and F, respectively, both compatible with $\check{\lambda}$, is a well-defined group homomorphism called the **twisted Chern character**.

Before proving Proposition 3.27, we recall the following lemma and its generalizations, which are certainly well-known. We include a proof here for sake of completeness. (See also Bunke and Nikolaus [6], Section 7).

Lemma 3.28. Suppose $U(1)$-gerbes λ and λ' defined on an open cover $\mathcal{U}$ of X are isomorphic: $\lambda'_{kji} = \lambda_{kji} + (\delta\chi)_{kji}$. Let $(\{A_{ji}\}, \{B_i\})$ on λ and $(\{A'_{ji}\}, \{B'_i\})$ on λ' be arbitrarily choosen connections. Then there exists Deligne 1-cochain $\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\})$ and $\xi \in \Omega^2(X; i\mathbb{R})$ such that $\check{\lambda}' = \check{\lambda}_\xi + D\check{\alpha}$, where $\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ and $\check{\lambda}' = (\{\lambda'_{kji}\}, \{A'_{ji}\}, \{B'_i\})$.

We need the following lemma, which is well-known.

Lemma 3.29. $\check{H}^p(\mathcal{U}, \Omega^q) = 0$, for all $p \geq 1$.

Proof of Lemma 3.29. See Bott and Tu [2], Proposition 8.5 in p.94. □

Proof of Lemma 3.28. We denote the 3-curvature of $\check{\lambda}$ and $\check{\lambda}'$ by H and H', respectively. The curvature 3-form of $\check{\lambda}$ and $\check{\lambda}'$ are cohomologous, i.e. $H' - H = d\zeta$ for some $\zeta \in \Omega^2(X; i\mathbb{R})$. Over each open set U_i, we have $d(B'_i - B_i) = d\zeta_i$, where $\zeta_i := \zeta|_{U_i}$, and by Poincaré's Lemma, there exists a 1-form ω_i on each U_i such that $B'_i = B_i + \zeta_i + d\omega_i$. Now by the cocycle condition, $dA'_{ji} = B'_j - B'_i = B_j - B_i + d(\omega_j - \omega_i) = dA_{ji} + d(\omega_j - \omega_i)$ and so there exists a $U(1)$-valued function μ_{ji} over each U_{ji} such that $A'_{ji} = A_{ji} + \omega_j - \omega_i + d\log\mu_{ji}$. Take the Čech differential of both sides and get $\delta(d\log(\chi\mu^{-1}))_{kji} = 0$. By Lemma 3.29, $d\log(\chi_{ji}\mu_{ji}^{-1}) = \gamma_j - \gamma_i$ for some $\gamma \in \check{C}^1(\mathcal{U}, \Omega^1)$, hence $d\log(\chi_{ji}) = d\log(\mu_{ji}) + (\gamma_j - \gamma_i)$. Notice that the family $\{d\gamma_i\}_{i\in\Lambda}$ defines a global 2-form on X. Thus setting $\check{\alpha} = (\{\chi_{ji}\}, \{\omega_i - \gamma_i\})$ and $\xi|_{U_i} = \zeta_i + d\gamma_i$ proves the claim. □

Remark 3.30. When the underlying gerbes λ and λ' are identical, a special case of Lemma 3.28 indicates that, under different choices of connection on λ, the corresponding twisted Chern characters are related by exp of a global 2-form.

Proof of Proposition 3.27. Well-definedness: Let $\check{\lambda}$ be fixed. The image of ch is independent of choice of connections on twisted vector bundles E and F by Proposition 3.24.

Suppose $\check{\lambda}$ and $\check{\lambda}'$ are the same $U(1)$-gerbes λ endowed with different connections and $\check{\lambda}' = \check{\lambda} + D\check{\alpha}$. Then by Proposition 3.14, the image of ch is invariant under cohomologous change of $U(1)$-gerbe connection. (Notice that the image of ch is not invariant under the arbitrary change of $U(1)$-gerbe connection. See Remark 3.30.)

Suppose there exists a λ-twisted vector bundle G with an isomorphism $\varphi : E \oplus G \to \overline{E} \oplus G$. Let Γ^G be an arbitrary connection on G. By Lemma

3.8 and Proposition 3.12,

$$\begin{aligned}\mathrm{ch}(\Gamma^E)t{+}[\mathrm{ch}(\Gamma^G)] &= [\mathrm{ch}(\Gamma^E \oplus \Gamma^G)] \\ &= [\mathrm{ch}(\varphi^*(\Gamma^{\overline{E}} \oplus \Gamma^G))] \\ &= [\mathrm{ch}(\Gamma^{\overline{E}} \oplus \Gamma^G)] \\ &= [\mathrm{ch}(\Gamma^{\overline{E}})] + [\mathrm{ch}(\Gamma^G)].\end{aligned}$$

From this, well-definedness of ch on $K^0_{\mathrm{tor}}(\mathcal{U}, \lambda)$ follows.

Group homomorphism: This follows from Lemma 3.8.

$$\begin{aligned}\mathrm{ch}([E] - [F] + [\overline{E}] - [\overline{F}]) &= \mathrm{ch}([E \oplus \overline{E}] - [F \oplus \overline{F}]) \\ &= \mathrm{ch}(\Gamma^E \oplus \Gamma^{\overline{E}}) - \mathrm{ch}(\Gamma^F \oplus \Gamma^{\overline{F}}) \\ &= \mathrm{ch}(\Gamma^E) - \mathrm{ch}(\Gamma^F) + \mathrm{ch}(\Gamma^{\overline{E}}) - \mathrm{ch}(\Gamma^{\overline{F}}) \\ &= \mathrm{ch}([E] - [F]) + \mathrm{ch}([\overline{E}] - [\overline{F}]).\end{aligned}$$

□

Proposition 3.31. Let $\check{\lambda}$ be as above, E be a λ-twisted vector bundle $(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$, and Γ_0 and Γ_1 be two connections on E joined by two different paths of connections α_t and γ_t on E, such that each of α_t and γ_t is compatible with $\check{\lambda}$ for all $t \in I$. Then

$$\mathrm{cs}(\gamma) - \mathrm{cs}(\alpha) \in \mathrm{Im}(d + H).$$

Proof. The paths α and γ define connections on p^*E over $X \times I$, which we denote by $\widetilde{\alpha}$ and $\widetilde{\gamma}$, respectively. Then there exists a path of connections on p^*E interpolating between $\widetilde{\alpha}$ and $\widetilde{\gamma}$ (by Corollary 3.19). Accordingly this path of connection defines a connection $\widetilde{\beta}$ on q^*p^*E over $X \times I \times I$, where $q : X \times I \times I \to X \times I$ is the projection map forgetting the third factor. By applying Lemma 3.23 to the twisted Chern character form $\mathrm{ch}(q^*p^*E, \widetilde{\widetilde{\beta}})$, we get

$$\begin{aligned} d \int_{I \times I} \mathrm{ch}(q^*p^*E, \widetilde{\beta}) &= \int_{I \times I} d\mathrm{ch}(q^*p^*E, \widetilde{\beta}) + \int_{\partial(I \times I)} \mathrm{ch}(q^*p^*E, \widetilde{\beta}) \\ &= -\Big(\int_{I \times I} \mathrm{ch}(q^*p^*E, \widetilde{\beta}) \Big) \wedge H + \int_I \mathrm{ch}(p^*E, \widetilde{\gamma}) \\ &\quad - \int_I \mathrm{ch}(p^*E, \widetilde{\alpha}) \end{aligned}$$

Hence,

$$\mathrm{cs}(\gamma) - \mathrm{cs}(\alpha) = (d + H) \int_{I \times I} \mathrm{ch}(q^*p^*E, \widetilde{\beta})$$

$\square$

Proposition 3.32. Let $\varphi : E \to F$ be an isomorphism of λ-twisted vector bundles over X. Let $\gamma : t \mapsto \Gamma^t$ be a path of connections on F. Then $\mathrm{cs}(\varphi^*\gamma) = \mathrm{cs}(\gamma)$.

Proof. By Definition 3.22,

$$\mathrm{cs}(\varphi^*\gamma) = \int_I \mathrm{ch}(\widetilde{\varphi^*\gamma}) \tag{3.3}$$

where $\widetilde{\varphi^*\gamma}$ is a connection on p^*E defined locally by

$$\widetilde{\varphi^*\gamma}_i(x,t)(v) := \varphi^*\gamma(t)_i(x)(p_*v)$$

$$= \varphi_i(x)^{-1}\gamma(t)_i(x)(p_*v)\varphi(x) + \varphi(x)^{-1}d\varphi(x)(p_*v),$$

and $v \in T_{(x,t)(X\times I)}$. Define $\widehat{\gamma}(x,t) := p^*\gamma(t)(x,t)$.

It follows that $\mathrm{ch}(\widetilde{\varphi^*\gamma}) = \mathrm{ch}(\widehat{\gamma})$ by a similar calculation in the proof of Proposition 3.12. Hence the RHS of (3.3) becomes $\int_I \mathrm{ch}(\widehat{\gamma}) = \mathrm{cs}(\gamma)$. □

Chapter 4

Twisted differential K-theory

This chapter constitutes the main result of this thesis. We define differential twists and construct a twisted differential K-group (Sections 4.1, 4.2) using triples consisting of a twisted vector bundle, a connection, and an odd differential form modulo exact forms in a twisted de Rham complex. We verify that our construction is functorial (Section 4.3) and natural with respect to change of twists (Section 4.4). In sections 4.5 and 4.6, we define the I, R, and a maps and verify the exact sequence involving the a and I maps. Finally, we show commutativity of diagrams and exactness of sequences consisting the hexagon diagram à la Simons and Sullivan [35] (Section 4.7), and verify that maps I, R, and a are compatible with change of twists (Section 4.8).

4.1 Differential twists

Definition 4.1. The **torsion differential K-twists** for an open cover $\mathcal{U}$ of X, denoted by $\texttt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$, is a groupoid whose objects are $U(1)$-gerbes with connection $\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ each of which has an underlying $U(1)$-gerbe representing a torsion class in $H^3(X;\mathbb{Z})$. A morphism in this groupoid from $\check{\lambda}_1$ to $\check{\lambda}_2$, if it exists, consists of a Deligne 1-cochain $\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\}) \in \check{C}^1(\mathcal{U}, \Omega^1)$, a global 2-form $\xi \in \Omega^2(X; i\mathbb{R})$, and a composition of $\check{\alpha}$ and ξ such that

- $\check{\lambda}_1 \xrightarrow{\check{\alpha}} \check{\lambda}_2$, where $\check{\lambda}_2 = \check{\lambda}_1 + D\check{\alpha}$.
- $\check{\lambda}_1 = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \xrightarrow{\xi} \check{\lambda}_2 = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i + \xi_i\})$, where $\xi_i := \xi|_{U_i}$. We shall use an expression $\check{\lambda}_2 = \check{\lambda}_1 + \xi$.

Remark 4.2. The Hom set $\mathrm{Hom}(\check{\lambda}_1, \check{\lambda}_2)$ is defined by

$$\{(\check{\alpha}, \xi) \in \check{C}^1(\mathcal{U}; \Omega^0) \oplus \check{C}^0(\mathcal{U}; \Omega^1) \oplus \Omega^2(X; i\mathbb{R})) : \check{\lambda}_2 = \check{\lambda}_1 + D\check{\alpha} + \xi\}.$$

Proposition 4.3 (Existence)**.** Given any manifold X with an open cover $\mathcal{U}$, the torsion differential twist $\texttt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$ consists of at least one object.

Proof. The statement amounts to the existence of connection on a local $U(1)$-bundle gerbe, which follows from the existence of partitions of unity as shown in Murray [30]. $\square$

Notation 4.4. (1) The *torsion topological K-twists* for an open cover $\mathcal{U}$ of a manifold X is a groupoid, denoted by $\mathtt{Twist}_K^{\mathrm{tor}}(\mathcal{U})$, whose objects are $U(1)$-gerbes $\lambda = \{\lambda_{kji}\}$ each representing a torsion class in $H^3(X;\mathbb{Z})$. A morphism from λ_1 to λ_2 is a Čech 1-cochain $\alpha = (\chi_{ji}) \in C^1(\mathcal{U}, U(1))$ such that $\lambda_1 = \lambda_2 + \delta\alpha$.

(2) Define the groupoid $\Omega^3_{\mathrm{cl}}(X; i\mathbb{R})$ of $i\mathbb{R}$-valued closed differential 3-forms on X as follows. Objects are $i\mathbb{R}$-valued closed differential 3-forms on X. A morphism from ω to ω' is a differential 2-form α on X modulo exact forms satisfying $\omega = \omega' + d\alpha$.

Definition 4.5. The **forgetful** and **curvature** functors are given by the assignments

$$\mathcal{F} : \mathtt{Twist}_{\check{K}}^{\mathrm{tor}}(\mathcal{U}) \to \mathtt{Twist}_K^{\mathrm{tor}}(\mathcal{U})$$

$$\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \mapsto \{\lambda_{kji}\}$$

$$\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\}) \mapsto \alpha = (\{\chi_{ji}\})$$

$$\mathrm{Curv} : \mathtt{Twist}_{\check{K}}^{\mathrm{tor}}(\mathcal{U}) \to \Omega^3_{\mathrm{cl}}(X; i\mathbb{R})$$

$$\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \mapsto \mathrm{Curv}(\check{\lambda}) = H$$

$$\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\}) \mapsto 0 \mod \mathrm{Im}(d)$$

where $H|_{U_i} = dB_i$ for all $i \in \Lambda$.

Remark 4.6. Let $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ be a map with $\mathcal{V} = f^{-1}\mathcal{U}$. The following diagrams commute:

$$\begin{array}{ccc} \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{V}) & \xrightarrow{\mathcal{F}} & \mathtt{Twist}^{\mathrm{tor}}_{K}(\mathcal{V}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U}) & \xrightarrow{\mathcal{F}} & \mathtt{Twist}^{\mathrm{tor}}_{K}(\mathcal{U}) \end{array} \qquad \begin{array}{ccc} \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{V}) & \xrightarrow{\mathrm{Curv}} & \Omega^3_{\mathrm{cl}}(Y; i\mathbb{R}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U}) & \xrightarrow{\mathrm{Curv}} & \Omega^3_{\mathrm{cl}}(X; i\mathbb{R}) \end{array}$$

Notation 4.7. Throughout this chapter, we shall use the notation $\check{\lambda}$ to denote a differential twist $(\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$, H for $\mathrm{Curv}(\check{\lambda})$, and λ for $\mathcal{F}(\check{\lambda})$.

4.2 Twisted differential K-group

Definition 4.8. A $\check{K}^0(\mathcal{U}; \check{\lambda})$**-generator** is a triple (E, Γ, ω) consisting of a λ-twisted vector bundle E defined on an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ on X, a connection Γ on E compatible with $\check{\lambda}$, and $\omega \in \Omega^{\mathrm{odd}}(X; \mathbb{C})/\mathrm{Im}(d + H)$ (compare to the definition of structured vector bundle of Simons and Sullivan [35]).

Definition 4.9. Let E be any λ-twisted vector bundle with a path of connections γ joining Γ_0 and Γ_1 and each connection on the path is compatible with $\check{\lambda}$. Define $\mathrm{CS}(\Gamma_0, \Gamma_1) := \mathrm{cs}(\gamma) \mod \mathrm{Im}(d + H)$.

Remark 4.10. By Proposition 3.31, Definition 4.9 is independent of the choice of path of connections. Furthermore, we have

$$\mathrm{CS}(\Gamma_0, \Gamma_1) + \mathrm{CS}(\Gamma_1, \Gamma_2) = \mathrm{CS}(\Gamma_0, \Gamma_2).$$

Definition 4.11. Two $\check{K}^0(\mathcal{U}; \check{\lambda})$-generators (E, Γ, ω) and (E', Γ', ω') are **equivalent** if there exists a λ-twisted vector bundle with connection (F, Γ^F) and a λ-twisted vector bundle isomorphism $\varphi = \{\varphi_i\}_{i \in \Lambda} : E \oplus F \to E' \oplus F$ such that $\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\Gamma' \oplus \Gamma^F)) = \omega - \omega'$.

Lemma 4.12. The relation between triples in Definition 4.11 is an equivalence relation.

Proof. The relation is reflexive since cs of a loop is $(d + H)$-exact. For symmetry, suppose (E, Γ, ω) and (E', Γ', ω') are equivalent, i.e., there exists a λ-twisted vector bundle with connection (F, Γ^F) whose connection is compatible with $\check{\lambda}$ such that there is an isomorphism of λ-twisted vector bundles $\varphi : E \oplus F \xrightarrow{\cong} E' \oplus F$, and $\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\Gamma' \oplus \Gamma^F)) = \omega - \omega'$. By Proposition 3.32,

$$\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\Gamma' \oplus \Gamma^F)) = \mathrm{CS}((\varphi^{-1})^*(\Gamma \oplus \Gamma^F), \Gamma' \oplus \Gamma^F).$$

This proves the symmetry. For transitivity, suppose (E, Γ, ω) is equivalent to (E', Γ', ω') and (E', Γ', ω') is equivalent to $(E'', \Gamma'', \omega'')$, i.e., there exists a λ-twisted vector bundle with connection (F, Γ^F) whose connection is compatible with $\check{\lambda}$ such that there is an isomorphism of λ-twisted vector bundles $\varphi : E \oplus F \stackrel{\cong}{\to} E' \oplus F$, and $\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\Gamma' \oplus \Gamma^F)) = \omega - \omega'$, and there exists a λ-twisted vector bundle with connection $(F', \Gamma^{F'})$ whose connection is compatible with $\check{\lambda}$ such that there is an isomorphism of λ-twisted vector bundles $\varphi' : E' \oplus F' \stackrel{\cong}{\to} E'' \oplus F'$, and $\mathrm{CS}(\Gamma' \oplus \Gamma^{F'}, \varphi'^*(\Gamma'' \oplus \Gamma^{F'})) = \omega' - \omega''$. Then by taking the λ-twisted vector bundle with connection $(F \oplus F', \Gamma^F \oplus \Gamma^{F'})$, the isomorphism of λ-twisted vector bundles $\psi : E \oplus F \oplus F' \to E'' \oplus F \oplus F'$ is defined by the composition

$$E \oplus F \oplus F' \xrightarrow{\varphi \oplus \mathbf{1}} E' \oplus F \oplus F' \xrightarrow{\mathbf{1} \oplus \sigma} E' \oplus F' \oplus F \xrightarrow{\varphi' \oplus \mathbf{1}} E'' \oplus F' \oplus F \xrightarrow{\mathbf{1} \oplus \sigma^{-1}} E'' \oplus F \oplus F',$$

each of which is an isomorphism, and σ is the canonical λ-twisted vector bundle isomorphism $F \oplus F' \to F' \oplus F$. Furthermore,

$$
\begin{aligned}
&\mathrm{CS}(\Gamma \oplus \Gamma^F \oplus \Gamma^{F'}, \psi^*(\Gamma'' \oplus \Gamma^F \oplus \Gamma^{F'}))\\
&= \mathrm{CS}(\Gamma \oplus \Gamma^F \oplus \Gamma^{F'}, (\varphi \oplus \mathbf{1})^*(\Gamma' \oplus \Gamma^F \oplus \Gamma^{F'}))\\
&\qquad + \mathrm{CS}((\varphi \oplus \mathbf{1})^*(\Gamma' \oplus \Gamma^F \oplus \Gamma^{F'}), \psi^*(\Gamma'' \oplus \Gamma^F \oplus \Gamma^{F'}))\text{by Remark 4.10}\\
&\stackrel{*}{=} \omega - \omega'\\
&\qquad + \mathrm{CS}(\Gamma' \oplus \Gamma^F \oplus \Gamma^{F'}, \big((\mathbf{1} \oplus \sigma^{-1}) \circ (\varphi' \oplus \mathbf{1}) \circ (\mathbf{1} \oplus \sigma)\big)^*(\Gamma'' \oplus \Gamma^F \oplus \Gamma^{F'}))\\
&\stackrel{**}{=} \omega - \omega' + \mathrm{CS}(\Gamma' \oplus \Gamma^{F'} \oplus \Gamma^F, \big((\mathbf{1} \oplus \sigma^{-1}) \circ (\varphi' \oplus \mathbf{1})\big)^*(\Gamma'' \oplus \Gamma^F \oplus \Gamma^{F'}))\\
&= \omega - \omega' + \mathrm{CS}(\Gamma' \oplus \Gamma^{F'} \oplus \Gamma^F, (\varphi' \oplus \mathbf{1})^*(\Gamma'' \oplus \Gamma^{F'} \oplus \Gamma^F))\\
&= \omega - \omega' + \omega' - \omega'' = \omega - \omega'',
\end{aligned}
$$

At $*$ and $**$, we have used Proposition 3.32 for the twisted bundle isomorphism $(\varphi \oplus \mathbf{1})^{-1}$ and $(\mathbf{1} \oplus \sigma^{-1})^{-1}$, respectively. Hence (E, Γ, ω) is equivalent to $(E'', \Gamma'', \omega'')$. □

Lemma 4.13. Let $[(E, \Gamma^E, \omega)]$ and $[(F, \Gamma^F, \eta)]$ be equivalence classes of $K^0(\mathcal{U}; \check{\lambda})$-generators. The equivalence class of the $K^0(\mathcal{U}; \check{\lambda})$-generator $(E \oplus F, \Gamma^E \oplus \Gamma^F, \omega + \eta)$ is independent of the choice of representatives of $[(E, \Gamma^E, \omega)]$ and $[(F, \Gamma^F, \eta)]$.

Proof. It suffices to show that if two $\check{K}^0(\mathcal{U}; \check{\lambda})$-generators (E, Γ^E, ω) and (F, Γ^F, η) are equivalent, then $(E \oplus F, \Gamma^E \oplus \Gamma^F, \omega + \eta)$ is equivalent to $(\widetilde{E} \oplus F, \Gamma^{\widetilde{E}} \oplus \Gamma^F, \widetilde{\omega} + \eta)$. Suppose there exists a λ-twisted bundle with connection (G, Γ^G) whose connection is compatible with $\check{\lambda}$ and a λ-twisted vector bundle isomorphism $\varphi : E \oplus G \to \widetilde{E} \oplus G$ such that $\mathrm{CS}(\Gamma^E \oplus \Gamma^G, \varphi^*(\Gamma^{\widetilde{E}} \oplus \Gamma^G)) = \omega - \widetilde{\omega}$. Then there exists a λ-twisted vector bundle with connection (G, ∇^G) whose connection is compatible with $\check{\lambda}$ and a λ-twisted vector bundle isomorphism $\varphi \oplus \mathbf{1}_F : E \oplus G \oplus F \to \widetilde{E} \oplus G \oplus F$ such that $\mathrm{CS}(\Gamma^E \oplus \Gamma^G \oplus \Gamma^F, \varphi^*(\Gamma^{\widetilde{E}} \oplus \Gamma^G) \oplus \Gamma^F) = \omega - \widetilde{\omega} = \omega + \eta - (\widetilde{\omega} + \eta)$. Hence $(E \oplus F, \Gamma^E \oplus \Gamma^F, \omega + \eta)$ is equivalent to $(\widetilde{E} \oplus F, \Gamma^{\widetilde{E}} \oplus \Gamma^F, \widetilde{\omega} + \eta)$. □

Definition 4.14. The **addition** $+$ between two equivalence classes of $\check{K}^0(\mathcal{U}; \check{\lambda})$-generators is defined by $[(E, \Gamma^E, \omega)] + [(F, \Gamma^F, \eta)] := [(E \oplus F, \Gamma^E \oplus \Gamma^F, \omega + \eta)]$.

Hence the set of all equivalence classes of $\check{K}^0(\mathcal{U}; \check{\lambda})$-generators forms a commutative semigroup $(\mathfrak{G}, +)$.

Definition 4.15. Let $\check{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$. The **twisted differential K-group** is

$$\check{K}^0(\mathcal{U}, \check{\lambda}) := K(\mathfrak{G}),$$

where K denotes the group completion functor from commutative semigroups to abelian groups.

4.3 Functoriality

Lemma 4.16. Let $\check{\lambda}$ be a differential twist, $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ a λ-twisted vector bundle of rank n, and $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ connection on E compatible with $\check{\lambda}$. Also let $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ be a map with $\mathcal{V} = f^{-1}\mathcal{U}$. If two triples (E, Γ, ω) and (E', Γ', ω') are equivalent, then $(f^*E, f^*\Gamma, f^*\omega)$ and $(f^*E', f^*\Gamma', f^*\omega')$ are equivalent.

Proof. Suppose there exists a λ-twisted vector bundle with connection (F, Γ^F) compatible with $\check{\lambda}$ such that there is an isomorphism of λ-twisted vector bundles $\varphi : E \oplus F \overset{\cong}{\to} E' \oplus F$ and $\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\Gamma' \oplus \Gamma^F)) = \omega - \omega'$. We want to show that there exists a $f^*\lambda$-twisted vector bundle F' with connection $\Gamma^{F'}$ compatible with $f^*\check{\lambda}$ such that there is an isomorphism of $f^*\lambda$-twisted vector bundles $\varphi' : f^*E \oplus F' \overset{\cong}{\to} f^*E' \oplus F'$ and

$$\mathrm{CS}(f^*\Gamma \oplus \Gamma^{F'}, \varphi'^*(f^*\Gamma' \oplus \Gamma^{F'})) = f^*\omega - f^*\omega'. \tag{4.1}$$

We take

$$F' := f^*F, \qquad \Gamma^{F'} := f^*\Gamma^F, \qquad \varphi' := \varphi \circ f.$$

Note that the isomorphism φ is a family of maps $\varphi_i : U_i \to U(n)$ for $n = \mathrm{rank}(E \oplus F)$. By $\varphi \circ f$, we mean the family $\{\varphi_i \circ f\}_{i\in\Lambda}$. With these data, we verify (4.1). First consider a path of connections γ joining $\Gamma \oplus \Gamma^F$ and

$\varphi^*(\Gamma' \oplus \Gamma^F)$. The path γ defines a connection $\widetilde{\gamma}$ on $p^*(E \oplus F)$ by $\widetilde{\gamma}_i(x,t) := p^*\gamma(t)_i(x,t)$. By pulling it back via $(f \times \mathbf{1})$, we obtain a connection $(f \times \mathbf{1})^*\widetilde{\gamma}$ on $(f \times \mathbf{1})^* p^*(E \oplus F)$ over $Y \times I$. Now we consider a path of connections $f^*\gamma$ joining $f^*\Gamma \oplus f^*\Gamma^F$ and $(\varphi \circ f)^*(f^*\Gamma' \oplus f^*\Gamma^F)$ on $f^*E \oplus f^*F$. This path defines a connection $\widehat{\gamma}$ on $p^*(f^*E \oplus f^*F)$ by $\widehat{\gamma}(y,t) = p^* f^* \Gamma^t(y,t)$. We see that, for any $u \in T_{(y,t)}(Y \times I)$,

$$\begin{aligned}\widehat{\gamma}(y,t)(u) &= f^*\gamma(t)(y)(p_* u) = \gamma(t)(f(y))(f_* p_* u) \\ &= \gamma(t)(f(y))(p_*(f \times \mathbf{1})_* u) = \widetilde{\gamma}((f(y),t))((f \times \mathbf{1})_* u) \qquad (4.2) \\ &= (f \times \mathbf{1})^* \widetilde{\gamma}(y,t)(u).\end{aligned}$$

Now we verify that $f^* \int_I \mathrm{ch}(\widetilde{\gamma}) = \int_I \mathrm{ch}((f \times \mathbf{1})^* \widetilde{\gamma})$. Locally, we can write $\mathrm{ch}(\widetilde{\gamma})$ as

$$\mathrm{ch}(\widetilde{\gamma}) := a_I(x,t) dx^I + b_J(x,t) dx^J \wedge dt,$$

and from this obtain

$$\int_I \mathrm{ch}(\widetilde{\gamma}) = \Big(\int_0^1 b_J(x,t)dt\Big) dx^J.$$

We then see that

$$\begin{aligned}f^* \int_I \mathrm{ch}(\widetilde{\gamma}) &= \left(\int_0^1 b_J(f(x),t)dt\right) d(x \circ f)^J \\ &= \int_I \Big(a_I(f(x),t) d(x \circ f)^I + b_J(f(x),t) d(x \circ f)^J \wedge dt\Big) \\ &= \int_I (f \times \mathbf{1})^* \mathrm{ch}(\widetilde{\gamma}) = \int_I \mathrm{ch}((f \times \mathbf{1})^* \widetilde{\gamma}).\end{aligned}$$

From this and (4.2), the equation (4.1) follows. □

Proposition 4.17. Given a map $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ with $\mathcal{V} = f^{-1}\mathcal{U}$, the assignment

$$f^* : \check{K}^0(\mathcal{U}, \check{\lambda}) \to \check{K}^0(\mathcal{V}, f^*\check{\lambda})$$

$$[(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)] \mapsto [(f^*E, f^*\Gamma^E, f^*\omega)] - [(f^*F, f^*\Gamma^F, f^*\eta)]$$

is a well-defined group homomorphism.

Proof. <u>Well-definedness:</u> Suppose $[(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)] = [(\overline{E}, \Gamma^{\overline{E}}, \overline{\omega})] - [(\overline{F}, \Gamma^{\overline{F}}, \overline{\eta})]$. Equivalently,

$$[(E \oplus \overline{F}, \Gamma^E \oplus \Gamma^{\overline{F}}, \omega + \overline{\eta})] = [(\overline{E} \oplus F, \Gamma^{\overline{E}} \oplus \Gamma^F, \overline{\omega} + \eta)].$$

Equivalently, there exists another triple (G, Γ^G, μ) such that $(E \oplus \overline{F} \oplus G, \Gamma^E \oplus \Gamma^{\overline{F}} \oplus \Gamma^G, \omega + \overline{\eta} + \mu) \sim (\overline{E} \oplus F \oplus G, \Gamma^{\overline{E}} \oplus \Gamma^F \oplus G, \overline{\omega} + \eta + \mu)$. By Lemma 4.16, it follows that $(f^*E \oplus f^*\overline{F} \oplus f^*G, f^*\Gamma^E \oplus f^*\Gamma^{\overline{F}} \oplus f^*\Gamma^G, f^*\omega + f^*\overline{\eta} + f^*\mu) \sim (f^*\overline{E} \oplus f^*F \oplus f^*G, f^*\Gamma^{\overline{E}} \oplus f^*\Gamma^F \oplus f^*G, f^*\overline{\omega} + f^*\eta + f^*\mu)$, and hence

$$\begin{aligned}[(f^*E, f^*\Gamma^E, f^*\omega)] - [(f^*F, f^*\Gamma^F, f^*\eta)]\\ = [(f^*\overline{E}, f^*\Gamma^{\overline{E}}, f^*\overline{\omega})] - [(f^*\overline{F}, f^*\Gamma^{\overline{F}}, f^*\overline{\eta})].\end{aligned}$$

Group homomorphism:

$$
\begin{aligned}
& f^*\big([(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)] + [(\overline{E},\Gamma^{\overline{E}},\overline{\omega})] - [(\overline{F},\Gamma^{\overline{F}},\overline{\eta})]\big) \\
&= f^*\big([(E\oplus\overline{E},\Gamma^E\oplus\Gamma^{\overline{E}},\omega+\overline{\omega})] - [(F\oplus\overline{F},\Gamma^F\oplus\Gamma^{\overline{F}},\eta+\overline{\eta})]\big) \\
&= [(f^*E\oplus f^*\overline{E}, f^*\Gamma^E\oplus f^*\Gamma^{\overline{E}}, f^*\omega+f^*\overline{\omega})] \\
&\quad - [(f^*F\oplus f^*\overline{F}, f^*\Gamma^F\oplus f^*\Gamma^{\overline{F}}, f^*\eta+f^*\overline{\eta})] \\
&= [(f^*E, f^*\Gamma^E, f^*\omega)] - [(f^*F, f^*\Gamma^F, f^*\eta)] \\
&\quad + [(f^*\overline{E}, f^*\Gamma^{\overline{E}}, f^*\overline{\omega})] - [(f^*\overline{F}, f^*\Gamma^{\overline{F}}, f^*\overline{\eta})] \\
&= f^*\big([(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)]\big) + f^*\big([(\overline{E},\Gamma^{\overline{E}},\overline{\omega})] - [(\overline{F},\Gamma^{\overline{F}},\overline{\eta})]\big).
\end{aligned}
$$

□

Let **Man** be a category whose objects are connected compact smooth manifolds equipped with an open cover. A morphism from $(Y,\mathcal{V})$ to $(X,\mathcal{U})$ is a smooth map satisfying $\mathcal{V} = f^{-1}(\mathcal{U})$. Also let **Ab** be the category of abelian groups.

Corollary 4.18. $\check{K}^0(-,\check{\lambda}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Ab}$ is a functor.

4.4 Naturality of twists

Proposition 4.19. Let

$$\check{\lambda} = (\{\lambda_{kji}\},\{A_{ji}\},\{B_i\})\text{and}\check{\lambda}' = (\{\lambda'_{kji}\},\{A'_{ji}\},\{B'_i\})$$

be any two objects of $\mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$ satisfying that $\check{\lambda}' = \check{\lambda} + D\check{\alpha}$ for some $\check{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\})$. Let $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle of rank n and $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ a connection on E compatible with $\check{\lambda}$. Define:

$$\begin{aligned} E' &:= (\mathcal{U}, \chi_{ji}g_{ji}, \lambda'_{kji}) \\ \Gamma' &:= \{\Gamma'_i\}_{i\in\Lambda} \text{ where } \Gamma'_i := \Gamma_i + \Pi_i \cdot \mathbf{1} \\ \omega' &:= \omega \end{aligned} \tag{4.3}$$

Then the assignment

$$\phi_{\check{\alpha}} : \check{K}^0(\mathcal{U}; \check{\lambda}) \xrightarrow{\cong} \check{K}^0(\mathcal{U}; \check{\lambda}')$$

$$[(E, \Gamma, \omega)] - [(F, \nabla, \eta)] \mapsto [(E', \Gamma', \omega')] - [(F', \nabla', \eta')]$$

is an induced group isomorphism that is natural in $\mathcal{U}$.

Remark 4.20. The family Γ' above is a connection on a λ'-twisted vector bundle E' compatible with $\check{\lambda}'$:

$$\begin{aligned} &g_{ji}^{-1}\chi_{ji}^{-1}\Gamma'_j\chi_{ji}g_{ji} + g_{ji}^{-1}\chi_{ji}^{-1}d(\chi_{ji}g_{ji}) - A'_{ji} \cdot \mathbf{1} \\ &= g_{ji}^{-1}\Gamma_j g_{ji} + \Pi_j \cdot \mathbf{1} + \chi_{ji}^{-1}d\chi_{ji} \cdot \mathbf{1} + g_{ji}^{-1}dg_{ji} \\ &- (A_{ji} + \Pi_j - \Pi_i + d\log\chi_{ji}) \cdot \mathbf{1} \\ &= \Gamma_i + \Pi_i \cdot \mathbf{1} = \Gamma'_i. \end{aligned}$$

Proof of Proposition 4.19. Suppose $(E, \Gamma, \omega) \sim (\overline{E}, \overline{\Gamma}, \overline{\omega})$, i.e., there exists a twisted vector bundle F and a connection Γ^F compatible with $\check{\lambda}$ and a

λ-twisted vector bundle isomorphism $\varphi = \{\varphi_i\} : E \oplus F \to \overline{E} \oplus F$ such that

$$\mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\overline{\Gamma} \oplus \Gamma^F)) = \omega - \overline{\omega}.$$

We verify that (E', Γ', ω') and $(\overline{E}', \overline{\Gamma}', \overline{\omega}')$ are equivalent so that well-definedness of the map follows. We take a λ'-twisted vector bundle F' and a connection Γ'^F compatible with $\check{\lambda}'$ by applying the same rule in (4.3) to (F, Γ^F). There exists a λ'-twisted vector bundle isomorphism $\varphi = \{\varphi_i\} : E' \oplus F' \to \overline{E}' \oplus F'$ defined exactly the same as the above φ.[1] We have to show that

$$\mathrm{CS}(\Gamma' \oplus \Gamma'^F, \varphi^*(\overline{\Gamma}' \oplus \Gamma'^F)) = \omega' - \overline{\omega}'.$$

Suppose $\widetilde{\Gamma}$ is a connection on $p^*(E \oplus F)$ over $X \times I$ defined by a path of connections joining $\Gamma \oplus \Gamma^F$ and $\varphi^*(\overline{\Gamma} \oplus \Gamma^F)$ on $E \oplus F$ over X. By definition,

$$\mathrm{cs}(\Gamma^t) := \int_I \mathrm{ch}(\widetilde{\Gamma}) = \int_I \mathrm{rank}(E \oplus F) + \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{tr}(\widetilde{R}_i - p^* B_i \cdot \mathbf{1})^m.$$

[1] Let $\overline{g}_{ji}$, h_{ji} be transition maps of $\overline{E}$ and F, respectively. Since φ is an isomorphism, we have

$$\varphi_j(x)\big(\overline{g}_{ji}(x) \oplus h_{ji}(x)\big) = \big(\overline{g}_{ji}(x) \oplus h_{ji}(x)\big)\varphi_i(x)$$

for all $x \in U_{ij}$. From this we have

$$\varphi_j(x)\big(\overline{g}_{ji}(x)\chi_{ji}(x) \oplus h_{ji}(x)\chi_{ji}(x)\big) = \big(\overline{g}_{ji}(x)\chi_{ji}(x) \oplus h_{ji}(x)\chi_{ji}(x)\big)\varphi_i(x)$$

for all $x \in U_{ij}$.

We see that:

$$\begin{aligned}
\mathrm{CS}(\Gamma' \oplus \Gamma'^F, \varphi^*(\overline{\Gamma}' \oplus \Gamma'^F)) &= \mathrm{CS}(\Gamma \oplus \Gamma^F + \Pi \cdot \mathbf{1}, \varphi^*(\overline{\Gamma} \oplus \Gamma^F + \Pi \cdot \mathbf{1})) \\
&= \mathrm{CS}(\Gamma \oplus \Gamma^F + \Pi \cdot \mathbf{1}, \varphi^*(\overline{\Gamma} \oplus \Gamma^F) + \Pi \cdot \mathbf{1}) \\
&= \mathrm{cs}(\Gamma^t + \Pi \cdot \mathbf{1}) \mod \mathrm{Im}(d+H) \\
&= \int_I \mathrm{ch}(\widetilde{\Gamma} + p^*\Pi \cdot \mathbf{1}) \mod \mathrm{Im}(d+H).
\end{aligned} \tag{4.4}$$

Since

$$d\big(\widetilde{\Gamma}_i + p^*\Pi_i \cdot \mathbf{1}\big) + \big(\widetilde{\Gamma}_i + p^*\Pi_i \cdot \mathbf{1}\big) \wedge \big(\widetilde{\Gamma}_i + p^*\Pi_i \cdot \mathbf{1}\big) = \widetilde{R}_i + p^*d\Pi_i \cdot \mathbf{1},$$

we have

$$\begin{aligned}
\int_I \mathrm{ch}(\widetilde{\Gamma} + p^*\Pi \cdot \mathbf{1}) &= \int_I \mathrm{rank}(E' \oplus F') + \\
&\qquad \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{tr}(\widetilde{R}_i + p^*d\Pi_i \cdot \mathbf{1} - p^*B_i' \cdot \mathbf{1})^m \\
&= \int_I \mathrm{rank}(E \oplus F) + \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{tr}(\widetilde{R}_i - p^*B_i \cdot \mathbf{1})^m \\
&= \int_I \mathrm{ch}(\widetilde{\Gamma}).
\end{aligned}$$

Hence the far RHS of (4.4) is

$$\int_I \mathrm{ch}(\widetilde{\Gamma}) \Big/ \mathrm{Im}(d+H) = \mathrm{CS}(\Gamma \oplus \Gamma^F, \varphi^*(\overline{\Gamma} \oplus \Gamma^F)) = \omega - \overline{\omega}.$$

The map $\phi_{\breve{\alpha}}$ being one-to one and onto are obvious. It is a group homomorphism because

$$\phi_{\check{\alpha}}\big([(E_1,\Gamma_1,\omega_1)] - [(F_1,\nabla_1,\eta_1)] + [(E_2,\Gamma_2,\omega_2)] - [(F_2,\nabla_2,\eta_2)]\big)$$

$$= \phi_{\check{\alpha}}\big([(E_1\oplus E_2,\Gamma_1\oplus\Gamma_2,\omega_1+\omega_2)] - [(F_1\oplus F_2,\nabla_1\oplus\nabla_2,\eta_1+\eta_2)]\big)$$

$$= [(E_1'\oplus E_2',\Gamma_1'\oplus\Gamma_2',\omega_1'+\omega_2')] - [(F_1'\oplus F_2',\nabla_1'\oplus\nabla_2',\eta_1'+\eta_2')]$$

$$= [(E_1',\Gamma_1',\omega_1')] - [(F_1',\nabla_1',\eta_1')] + [(E_2',\Gamma_2',\omega_2')] - [(F_2',\nabla_2',\eta_2')]$$

$$= \phi_{\check{\alpha}}\big([(E_1,\Gamma_1,\omega_1)] - [(F_1,\nabla_1,\eta_1)]\big) + \phi_{\check{\alpha}}\big([(E_2,\Gamma_2,\omega_2)] - [(F_2,\nabla_2,\eta_2)]\big)$$

The map $\phi_{\check{\alpha}}$ is natural in $\mathcal{U}$. Suppose $f:(Y,\mathcal{V})\to(X,\mathcal{U})$ is a map with $\mathcal{V}:=f^{-1}(\mathcal{U})$. The map f induces an isomorphism $f^*\check{\alpha}=(\{f^*\chi_{ji}\},\{f^*\Pi_i\})$ between $f^*\check{\lambda}$ and $f^*\check{\lambda}'$ as well as an isomorphism $\phi_{f^*\check{\alpha}}:\check{K}^0(\mathcal{U};f^*\check{\lambda})\to\check{K}^0(\mathcal{V};f^*\check{\lambda}')$. Clearly $\phi_{f^*\check{\alpha}}\circ f^* = f^*\circ\phi_{\check{\alpha}}$. □

Recall Notation 3.16.

Proposition 4.21. Let $\check{\lambda}\in\mathtt{Twist}_{\check{K}}^{\mathrm{tor}}(\mathcal{U})$, and $\xi\in\Omega^2(X;i\mathbb{R})$. The assignment

$$\Xi:\check{K}^0(\mathcal{U};\check{\lambda})\overset{\cong}{\to}\check{K}^0(\mathcal{U};\check{\lambda}_\xi)$$

$$[(E,\Gamma,\omega)]-[(F,\nabla,\eta)]\mapsto[(E,\Gamma_\xi,\omega\wedge\exp(-\xi))]-[(F,\nabla_\xi,\eta\wedge\exp(-\xi))]$$

is a group isomorphism that is natural in $\mathcal{U}$.

Proof. We first show that, if $(E,\Gamma,\omega)\sim(\overline{E},\overline{\Gamma},\overline{\omega})$, then

$$(E,\Gamma_\xi,\omega\wedge\exp(-\xi))\sim(\overline{E},\overline{\Gamma}_\xi,\overline{\omega}\wedge\exp(-\xi)).$$

By the premise, there exists a λ-twisted vector bundle G and a connection Γ^G on G compatible with $\check{\lambda}$ and an isomorphism $\varphi : E \oplus G \to \overline{E} \oplus G$, such that $\omega - \overline{\omega} = \mathrm{CS}(\Gamma \oplus \Gamma^G, \varphi^*(\overline{\Gamma} \oplus \Gamma^G)) = \int_I \mathrm{ch}(\widetilde{\Gamma}) \mod \mathrm{Im}(d+H)$, where $\widetilde{\Gamma}$ is a connection on $p^*(E \oplus G)$ defined by pullback of connections on a straight line path joining $\Gamma \oplus \Gamma^G$ and $\varphi^*(\overline{\Gamma} \oplus \Gamma^G)$. Accordingly,

$$\begin{aligned}
\mathrm{CS}(\Gamma_\xi \oplus \Gamma^G_\xi, \varphi^*(\overline{\Gamma}_\xi \oplus \Gamma^G_\xi)) &= \int_I \left(\mathrm{ch}(\widetilde{\Gamma}) \wedge \exp(-p^*\xi)\right) \mod \mathrm{Im}(d + H + d\xi) \\
&= \left(\int_I \mathrm{ch}(\widetilde{\Gamma})\right) \wedge \exp(-\xi) \mod \mathrm{Im}(d + H + d\xi) \\
&= (\omega - \overline{\omega}) \wedge \exp(-\xi) \mod \mathrm{Im}(d + H + d\xi).
\end{aligned}$$

From this, well-definedness of the map Ξ follows. The map Ξ being one-to one, onto, and group homomorphism are all obvious. It is natural in $\mathcal{U}$. Suppose $f : (Y, \mathcal{V}) \to (X, \mathcal{U})$ is a map with $\mathcal{V} := f^{-1}(\mathcal{U})$. The map f induces an isomorphism $f^*\xi : f^*\check{\lambda} \to f^*\check{\lambda}_\xi$ as well as an isomorphism $f^*\Xi : \check{K}^0(\mathcal{U}; f^*\check{\lambda}) \to \check{K}^0(\mathcal{V}; f^*\check{\lambda}_\xi)$. Apparently $f^*\Xi \circ f^* = f^* \circ \Xi$. □

4.5 The I and R map

Proposition 4.22. Let $\check{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$. The assignment

$$I : \check{K}^0(\mathcal{U}, \check{\lambda}) \to K^0(\mathcal{U}, \lambda)$$

$$[(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)] \mapsto [E] - [F]$$

is a group homomorphism which is natural in $\mathcal{U}$.

Proof. Well-definedness: Suppose $[(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]=[(\overline{E},\Gamma^{\overline{E}},\overline{\omega})]-[(\overline{F},\Gamma^{\overline{F}},\overline{\eta})]$. There exist λ-twisted vector bundles G, H, and an isomorphism $\varphi: E\oplus\overline{F}\oplus G\oplus H\to\overline{E}\oplus F\oplus G\oplus H$. This means $[E\oplus\overline{F}]=[\overline{E}\oplus F]$, equivalently $[E]-[F]=[\overline{E}]-[\overline{F}]$.

Group homomorphism:

$I\big([(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]+[(\overline{E},\Gamma^{\overline{E}},\overline{\omega})]-[(\overline{F},\Gamma^{\overline{F}},\overline{\eta})]\big)=I\big([(E\oplus\overline{E},\Gamma^E\oplus\Gamma^{\overline{E}},\omega+\overline{\omega})]-[(F\oplus\overline{F},\Gamma^F\oplus\Gamma^{\overline{F}},\eta+\overline{\eta})]\big)=[E\oplus\overline{E}]-[F\oplus\overline{F}]=[E]-[F]+[\overline{E}]-[\overline{F}]$.

Naturality: Let $f:(Y,f^{-1}\mathcal{U})\to(X,\mathcal{U})$ be a map. For any $[(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]\in\check{K}^0(\mathcal{U},\check{\lambda})$,

$$\begin{aligned}&f^*\circ I_X([(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)])\\=&f^*([E]-[F])=[f^*E]-[f^*F]\\=&I_Y([(f^*E,f^*\Gamma^E,f^*\omega)]-[(f^*F,f^*\Gamma^F,f^*\eta)])\\=&I_Y\circ f^*([(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]).\end{aligned}$$

□

Proposition 4.23. Let $\check{\lambda}\in\texttt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$. The assignment

$$R:\check{K}^0(\mathcal{U},\check{\lambda})\to\Omega^{\mathrm{even}}(X;\mathbb{C})$$

$$[(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]\mapsto\mathrm{ch}(\Gamma^E)+(d+H)\omega-\mathrm{ch}(\Gamma^F)-(d+H)\eta$$

is a group homomorphism which is natural in $\mathcal{U}$.

Proof. Well-definedness: Suppose $[(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]=[(\overline{E},\Gamma^{\overline{E}},\overline{\omega})]-$

$[(\overline{F}, \Gamma^{\overline{F}}, \overline{\eta})]$. There exists another triple (G, Γ^G, μ) such that $(E \oplus \overline{F} \oplus G, \Gamma^E \oplus \Gamma^{\overline{F}} \oplus \Gamma^G, \omega + \overline{\eta} + \mu) \sim (\overline{E} \oplus F \oplus G, \Gamma^{\overline{E}} \oplus \Gamma^F \oplus G, \overline{\omega} + \eta + \mu)$. Equivalently, there exists a λ-twisted vector bundle with connection (H, Γ^H) and an isomorphism $\varphi : E \oplus \overline{F} \oplus G \oplus H \to \overline{E} \oplus F \oplus G \oplus H$ such that

$$\mathrm{CS}(\Gamma^E \oplus \Gamma^{\overline{F}} \oplus \Gamma^G \oplus \Gamma^H, \varphi^*(\Gamma^{\overline{E}} \oplus \Gamma^F \oplus \Gamma^G \oplus \Gamma^H)) = (\omega + \overline{\eta} + \mu) - (\overline{\omega} + \eta + \mu).$$

If we apply $(d + H)$ to both sides, we get

$$\begin{aligned} &\mathrm{ch}(\varphi^*(\Gamma^{\overline{E}} \oplus \Gamma^F \oplus \Gamma^G \oplus \Gamma^H)) - \mathrm{ch}(\Gamma^E \oplus \Gamma^{\overline{F}} \oplus \Gamma^G \oplus \Gamma^H) \\ =&\mathrm{ch}(\Gamma^{\overline{E}} \oplus \Gamma^F) - \mathrm{ch}(\Gamma^E \oplus \Gamma^{\overline{F}}) \\ =&(d + H)(\omega + \overline{\eta} - \overline{\omega} - \eta). \end{aligned}$$

Rewriting the last equality, we obtain

$$\mathrm{ch}(\Gamma^{\overline{E}}) - \mathrm{ch}(\Gamma^{\overline{F}}) + (d + H)(\overline{\omega} - \overline{\eta}) = \mathrm{ch}(\Gamma^E) - \mathrm{ch}(\Gamma^F) + (d + H)(\omega - \eta).$$

Group homomorphism:

$$\begin{aligned} &R\big([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)] + [(\overline{E}, \Gamma^{\overline{E}}, \overline{\omega})] - [(\overline{F}, \Gamma^{\overline{F}}, \overline{\eta})]\big) \\ =\ &R\big([(E \oplus \overline{E}, \Gamma^E \oplus \Gamma^{\overline{E}}, \omega + \overline{\omega})] - [(F \oplus \overline{F}, \Gamma^F \oplus \Gamma^{\overline{F}}, \eta + \overline{\eta})]\big) \\ =\ &\mathrm{ch}(\Gamma^E \oplus \Gamma^{\overline{E}}) + (d + H)(\omega + \overline{\omega}) - \mathrm{ch}(\Gamma^F \oplus \Gamma^{\overline{F}}) - (d + H)(\eta + \overline{\eta}) \\ =\ &\mathrm{ch}(\Gamma^E) + (d + H)\omega - \mathrm{ch}(\Gamma^F) - (d + H)\eta + \mathrm{ch}(\Gamma^{\overline{E}}) + (d + H)\overline{\omega} - \mathrm{ch}(\Gamma^{\overline{F}}) \\ &- (d + H)\overline{\eta}. \end{aligned}$$

Naturality: Let $f : (Y, f^{-1}\mathcal{U}) \to (X, \mathcal{U})$ be a map. For any $[(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)] \in \check{K}^0(\mathcal{U}, \check{\lambda})$,

$$\begin{aligned} & f^* \circ R_X([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]) \\ =& f^*(\mathrm{ch}(\Gamma^E) + (d+H)\omega - \mathrm{ch}(\Gamma^F) - (d+H)\eta) \\ =& \mathrm{ch}(f^*\Gamma^E) + (d + f^*H)f^*\omega - \mathrm{ch}(f^*\Gamma^F) - (d + f^*H)f^*\eta \\ =& R_Y([(f^*E, f^*\Gamma^E, f^*\omega)] - [(f^*F, f^*\Gamma^F, f^*\eta)]) \\ =& R_Y \circ f^*([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]). \end{aligned}$$

□

4.6 The a map and the exact sequence involving the a and I maps

Lemma 4.24. Let $\check{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$. There exists a differential twist $\check{\lambda}^\circ = (\{\lambda^\circ_{kji}\}, \{A^\circ_{ji}\}, \{B^\circ_i\})$ and an isomorphism $\check{\alpha} : \check{\lambda} \to \check{\lambda}^\circ$ such that $\check{\lambda} + D\check{\alpha} = \check{\lambda}^\circ$ with $\{\lambda^\circ_{kji}\}$ a family of constant maps and $A^\circ_{ji} = 0$ for all $i, j \in \Lambda$.

Proof. By Proposition A.3, we may assume that each λ_{kji} of $\check{\lambda}$ is constant. Since $\check{\lambda}$ is a cocycle, it follows that

$$d \log \lambda_{kji} = A_{ji} + A_{ik} + A_{kj} = \delta A_{kji}.$$

Since λ_{kji} is constant, $(A_{ji}) \in \check{C}^1(\mathcal{U}, \Omega^1)$ is a Čech 1-cocycle. Now by Lemma

3.29, for $p = q = 1$, there exists $(\Pi_i) \in \check{C}^0(\mathcal{U}, \Omega^1)$ such that

$$A_{ji} = \Pi_i - \Pi_j \qquad \text{for all } x \in U_{ij}.$$

Accordingly, if we take $\check{\alpha} = (\mathbf{1}, \Pi_i)$, we get:

$$\lambda^\circ_{kji} = \lambda_{kji}$$

$$A^\circ_{ji} = A_{ji} + \Pi_j - \Pi_i = 0$$

$$B^\circ_i = B_i + d\Pi_i.$$

□

Notation 4.25. We shall henceforth use the notation $\check{\lambda}^\circ$ to denote $(\{\lambda^\circ_{kji}\}, \{A^\circ_{ji}\}, \{B^\circ_i\}) \in \mathtt{Twist}^{\mathrm{tor}}_K(\mathcal{U})$ such that each λ°_{kji} is a constant map, and $A_{ji} = 0$ for every $i, j \in \Lambda$.

Lemma 4.26. Let $\check{\lambda}^\circ \in \mathtt{Twist}^{\mathrm{tor}}_K(\mathcal{U})$, and let $E = (\{g^\circ_{ji}\}, \{\lambda^\circ_{kji}\})$ be a λ°-twisted vector bundle of rank n such that each transition map g°_{ji} is a constant map. Then the family $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ with $\Gamma_i = O$ for all $i \in \Lambda$ is a connection on E compatible with $\check{\lambda}^\circ$. Here O is the zero matrix.

Proof. From the defining relation of a connection on a twisted vector bundle, $\Gamma_i = g_{ji}^{-1}\Gamma_j g_{ji} + g_{ji}^{-1}dg_{ji} - A^\circ_{ji}\cdot\mathbf{1}$. The existence of E follows from Proposition A.4. □

Notation 4.27. We denote the connection defined in Lemma 4.26 by $O =$

$\{O_i\}_{i\in\Lambda}$.

Notation 4.28. We refer the reader to Appendix B for the definition and properties of the odd twisted Chern character. In particular, for a fixed differential twist $\check{\lambda} \in \mathtt{Twist}_{\check{K}}^{\mathrm{tor}}(\mathcal{U})$ and a triple (E, ϕ, Γ^E) consisting of a λ-twisted vector bundle, an automorphism ϕ of E, and a connection Γ^E on E associated with $\check{\lambda}$, there is an *odd twisted Chern character form* $\mathrm{Ch}(E, \phi, \Gamma^E) := \mathrm{cs}(\gamma)$ where $\gamma(t) := (1-t)\Gamma^E + t\phi^*\Gamma^E$. We denote by $\mathrm{Im}(\mathrm{Ch})$ the abelian group generated by all odd twisted Chern character forms, and also denote by $\Omega_{H,\mathrm{Ch}}$ the abelian group $\mathrm{Im}(\mathrm{Ch}) + \mathrm{Im}(d + H)$.

Definition 4.29. Let $\check{\lambda}^\circ \in \mathtt{Twist}_{\check{K}}^{\mathrm{tor}}(\mathcal{U})$. Define:

$$a^\circ : \Omega^{\mathrm{odd}}(X; \mathbb{C})/\Omega_{H,\mathrm{Ch}} \to \check{K}^0(\mathcal{U}, \check{\lambda}^\circ)$$

$$\theta \mapsto [(T, O, \theta)] - [(T, O, 0)],$$

where $T = (\{g_{ji}\}, \{\lambda^\circ_{kji}\})$ is a λ°-twisted vector bundle of rank n for some $n \in \mathbb{N}$, such that each g_{ji} is a constant map.

Lemma 4.30. The map a° is well-defined.

Proof. By Lemma 4.26, there exists a connection O on twisted vector bundle T. The image of the a map does not depend on a particular choice of T. To see this, if S is another λ°-twisted vector bundle whose transition maps are all constant maps, then

$$\begin{aligned}
[(T,O,\theta)] - [(T,O,0)] &= [(T,O,\theta)] + [(S,O,0)] - [(S,O,0)] - [(T,O,0)] \\
&= [(T \oplus S, O \oplus O, \theta)] - [(S \oplus T, O \oplus O, 0)] \\
&= [(T,O,0)] + [(S,O,\theta)] - [(S,O,0)] - [(T,O,0)] \\
&= [(S,O,\theta)] - [(S,O,0)].
\end{aligned}$$

We also observe that

$$\begin{aligned}
a^\circ(\theta + \mathrm{Ch}(E,\phi,\Gamma^E)) = [(T,O,\theta + \mathrm{Ch}(E,\phi,\Gamma^E))] - [(T,O,0)] \\
=&[(T \oplus E, O \oplus \Gamma^E, \theta + \mathrm{cs}((1-t)\Gamma^E + t\phi^*\Gamma^E))] \\
&- [(E,\Gamma^E,0)] - [(T,O,0)] \\
=&[(T,O,\theta)] + [(E,\Gamma^E,\mathrm{cs}((1-t)\Gamma^E + t\phi^*\Gamma^E))] \\
&- [(E,\Gamma^E,0)] - [(T,O,0)] \\
=&[(T,O,\theta)] - [(T,O,0)] = a^\circ(\theta),
\end{aligned}$$

because $(E, \Gamma^E, \mathrm{cs}(t \mapsto (1-t)\Gamma^E + t\phi^*\Gamma^E))$ and $(E, \Gamma^E, 0)$ are equivalent as triples. □

Proposition 4.31. The map a° is a group homomorphism and is natural in $\mathcal{U}$.

Proof. Group homomorphism: For any $\theta, \zeta \in \Omega^{\text{odd}}(X;\mathbb{C})/\Omega_{H,\text{Ch}}$,

$$\begin{aligned} a^\circ(\theta+\zeta) &= [(T,O,\theta+\zeta)] - [(T,O,0)] \\ &= [(T\oplus T, O\oplus O, \theta+\zeta)] - [(T\oplus T, O\oplus O, 0)] \\ &= [(T,O,\theta)] - [(T,O,0)] + [(T,O,\zeta)] - [(T,O,0)] \\ &= a^\circ(\theta) + a^\circ(\zeta). \end{aligned}$$

Naturality: Let $f : (Y, f^{-1}\mathcal{U}) \to (X,\mathcal{U})$ be a map. For any $\theta \in \Omega^{\text{odd}}(X;\mathbb{C})/\Omega_{H,\text{Ch}}$,

$$\begin{aligned} f^* \circ a^\circ(\theta) &= f^*([(T,O,\theta)] - [(T,O,0)]) \\ &= [(f^*T, O, f^*\theta)] - [(f^*T, O, 0)]) = a^\circ \circ f^*(\theta), \end{aligned}$$

where T is a λ°-twisted vector bundle over X whose transition maps are constant maps. The pullback $(\lambda^\circ \circ f)$-twisted bundle $f^*(T)$ has all transition maps as constant maps. Since the image of a° does not depend on the choice of a twisted vector bundle with constant transition maps and because the odd twisted Chern character forms satisfy functoriality, the last equality follows. For the second equality, we have used the fact that f^*O is the connection O on f^*T. Notice that the pullback twisted vector bundle $f^*(T_X)$ admits the connection O because each $g_{ji} \circ f$ is a constant map and $f^*A^\circ_{ji} = 0$ for every $i, j \in \Lambda$. □

Proposition 4.32. The following sequence is exact:

$$0 \to \Omega^{\mathrm{odd}}(X;\mathbb{C})/\Omega_{H,\mathrm{Ch}} \xrightarrow{a^\circ} \check{K}^0(\mathcal{U},\check{\lambda}^\circ) \xrightarrow{I} K^0(\mathcal{U},\lambda^\circ) \to 0.$$

Proof. It is obvious that I is surjective, and Im $a \subseteq \ker I$. We show the other inclusion.

Let $[(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)] \in \check{K}^0(\mathcal{U},\check{\lambda}^\circ)$ and suppose $I\big([(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)]\big) = 0$. Then there exists a λ°-twisted vector bundle G and an isomorphism of λ°-twisted vector bundles $\varphi : E \oplus G \to F \oplus G$. Choose any connection Γ^G on G that is compatible with $\check{\lambda}^\circ$. Then

$$\begin{aligned}[(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)] &= [(E\oplus G,\Gamma^E\oplus\Gamma^G,\omega)] - [(F\oplus G,\Gamma^F\oplus\Gamma^G,\eta)]\\ &\overset{*}{=} [(E\oplus G,\Gamma^E\oplus\Gamma^G,\omega)] - [(E\oplus G,\Gamma^E\oplus\Gamma^G,\mu)],\end{aligned}$$

where $\mu := \eta + \mathrm{CS}(\Gamma^E\oplus\Gamma^G,\varphi^*(\Gamma^F\oplus\Gamma^G))$, and $*$ follows from the fact that $(E\oplus G,\Gamma^E\oplus\Gamma^G,\mu)$ is equivalent to $(F\oplus G,\Gamma^F\oplus\Gamma^G,\eta)$. We now add and subtract $(T,O,0)$, and get:

$$\begin{aligned}&[(E\oplus G,\Gamma^E\oplus\Gamma^G,\omega)] - [(E\oplus G,\Gamma^E\oplus\Gamma^G,\mu)]\\ &= [(E\oplus G\oplus T,\Gamma^E\oplus\Gamma^G\oplus O,\omega)] - [(E\oplus G\oplus T,\Gamma^E\oplus\Gamma^G\oplus O,\mu)]\\ &= [(E\oplus G,\Gamma^E\oplus\Gamma^G,0)] + [(T,O,\omega)]\\ &\qquad - [(E\oplus G,\Gamma^E\oplus\Gamma^G,0)] - [(T,O,\mu)]\\ &= [(T,O,\omega-\mu)] - [(T,O,0)] = a^\circ(\omega-\mu).\end{aligned}$$

We now show injectivity of the a map. Suppose

$$a(\theta) = [(T, O, \theta)] - [(T, O, 0)] = 0.$$

Equivalently, there exists a λ°-twisted vector bundle with connection (E, Γ^E) whose connection is compatible with $\check{\lambda}^\circ$ and an isomorphism $\varphi : T \oplus E \to T \oplus E$ satisfying that $\theta = \mathrm{CS}(O \oplus \Gamma^E, \varphi^*(O \oplus \Gamma^E)) = \mathrm{Ch}(T \oplus E, \varphi, O \oplus \Gamma^E)$. Hence the result. □

Now we define the a map into the twisted differetial K-group with an arbitraty differential twist $\check{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$.

Definition 4.33. Let $\check{\lambda}^\circ \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$ be isomorphic via $\check{\alpha}^\circ := (\{\chi_{ji}\}, \{\Pi_i\})$ to $\check{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\check{K}}(\mathcal{U})$. We define the map $a : \Omega^{\mathrm{odd}}(X; \mathbb{C})/\Omega_{H,\mathrm{Ch}} \to \check{K}^0(\mathcal{U}, \check{\lambda})$ by $\phi_{\check{\alpha}^\circ} \circ a^\circ$, i.e.,

$$a : \Omega^{\mathrm{odd}}(X; \mathbb{C})/\Omega_{H,\mathrm{Ch}} \to \check{K}^0(\mathcal{U}, \check{\lambda})$$

$$\theta \mapsto [(\phi_{\alpha^\circ} T, \{\Pi_i\}, \theta)] - [(\phi_{\alpha^\circ} T, \{\Pi_i\}, 0)],$$

where $\phi_{\alpha^\circ} T = (\mathcal{U}, \{\theta_{ji} g^\circ_{ji}\}, \{\lambda^\circ_{kji}\})$ and $T = (\mathcal{U}, \{g^\circ_{ji}\}, \{\lambda^\circ_{kji}\})$ are λ°-twisted vector bundles of rank n for some $n \in \mathbb{N}$, and each g°_{ji} is a constant map.

Remark 4.34. The a map in Definition 4.33 satisfies all properties that the a° map in Definition 4.29 satisfy.

4.7 The hexagon diagram

Notation 4.35. We denote by $\mathrm{Pr} : \Omega^{\mathrm{even}}(X;\mathbb{C})_{\mathrm{closed}} \to H_H^{\mathrm{even}}(X;\mathbb{C})$ the map taking twisted de Rham cohomology class, and $r : H_H^{\mathrm{odd}}(X;\mathbb{C}) \to \Omega^{\mathrm{odd}}(X;\mathbb{C})/\Omega_{H,\mathrm{Ch}}$ the map that sends an odd twisted de Rham cohomology class $[\omega]$ to $\omega + \Omega_{H,\mathrm{Ch}}$. The map r is well-defined by definition of $\omega + \Omega_{H,\mathrm{Ch}}$ (see Notation 4.28).

Throughout this section, we assume that $\check{\lambda} = \check{\lambda}^\circ + D\check{\alpha}^\circ$ where $\check{\alpha}^\circ = (\{\chi_{ji}\}, \{\Pi_i\})$. Also, ϕ_{α° is the map between twisted K-groups induced by the functor $\mathbf{Bun}(\mathcal{U}, \lambda^\circ) \to \mathbf{Bun}(\mathcal{U}, \lambda)$ that takes a λ°-twisted vector bundle $T = (\{g_{ji}^\circ\}, \{\lambda_{kji}^\circ\})$ to the λ-twisted vector bundle $\phi_{\alpha^\circ} T = (\mathcal{U}, \{\chi_{ji} g_{ji}^\circ\}, \{\lambda_{kji}\})$, and takes any morphism to itself.

Proposition 4.36. For the maps I, R, and a from or into $\check{K}^0(\mathcal{U};\check{\lambda})$, the following holds:

(1) $\mathrm{ch} \circ I = \mathrm{Pr} \circ R$.

(2) $R \circ a = d + H$.

Proof. (1)

$$
\begin{aligned}
&\mathrm{Pr}\circ R([(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)])\\
=&\mathrm{Pr}\big(\mathrm{ch}(\Gamma^E)+(d+H)\omega-\mathrm{ch}(\Gamma^F)-(d+H)\eta\big)\\
=&\mathrm{ch}(E)-\mathrm{ch}(F)\\
=&\mathrm{ch}\circ I([(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]).
\end{aligned}
$$

(2)

$$
\begin{aligned}
R\circ a(\theta)&=R([(\phi_{\alpha^\circ}T,\{\Pi_i\},\theta)]-[(\phi_{\alpha^\circ}T,\{\Pi_i\},0)])\\
&=\mathrm{ch}(\{\Pi_i\})+(d+H)\theta-\mathrm{ch}(\{\Pi_i\})=(d+H)(\theta+\Omega_{H,\mathrm{Ch}}).
\end{aligned}
$$

□

Definition 4.37. We define maps α and β as follows:

$$\alpha: H_H^{\mathrm{odd}}(X;\mathbb{C})\to\ker R$$

$$(E,\Gamma^E,\omega)-(F,\Gamma^F,\eta)\mapsto[E]-[F],$$

$$\beta:\ker R\to K^0(\mathcal{U},\lambda)$$

$$[\omega]\mapsto(\phi_\alpha T,\{\Pi_i\},\omega)-(\phi_\alpha T,\{\Pi_i\},0)$$

where T is a λ°-twisted vector bundle whose transition maps are constant maps.

Remark 4.38. The maps α and β are well-defined group homomorphisms.

Proposition 4.39. (1) $a\circ r=\mathrm{incl}\circ\alpha$.

(2) $\beta=I\circ\mathrm{incl}$.

(3) The following sequences are exact:

$$H_H^{\text{odd}}(X;\mathbb{C}) \xrightarrow{\alpha} \ker R \xrightarrow{\beta} K^0(\mathcal{U},\lambda) \xrightarrow{\text{ch}} H_H^{\text{even}}(X;\mathbb{C})$$

$$H_H^{\text{odd}}(X;\mathbb{C}) \xrightarrow{r} \Omega^{\text{odd}}(X;\mathbb{C})/\text{Im}(d+H) \xrightarrow{d+H} \Omega^{\text{even}}(X;\mathbb{C}) \xrightarrow{\text{Pr}} H_H^{\text{even}}(X;\mathbb{C})$$

Proof. All claims are obvious except that $\ker(\beta) \subseteq \text{Im}(\alpha)$, which we prove presently. Take an arbitrary element $[(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)] \in \ker R$ whose image under β is zero, i.e., there exists a λ-twisted vector bundle G defined on $\mathcal{U}$ and an isomorphism $\varphi : E \oplus G \xrightarrow{\cong} F \oplus G$. Choose any connection Γ^G on G that is compatible with $\check{\lambda}$. At this point we introduce the following notation:

$$\zeta := \text{CS}\left(\Gamma^E \oplus \Gamma^G, \varphi^*(\Gamma^F \oplus \Gamma^G)\right).$$

By adding and subtracting $[(G,\Gamma^G,-\eta-\zeta)]$,

$$\begin{aligned}&[(E,\Gamma^E,\omega)] - [(F,\Gamma^F,\eta)]\\ =&[(E\oplus G,\Gamma^E\oplus\Gamma^G,\omega-\eta-\zeta)] - [(F\oplus G,\Gamma^F\oplus\Gamma^G,-\zeta)].\end{aligned} \tag{4.5}$$

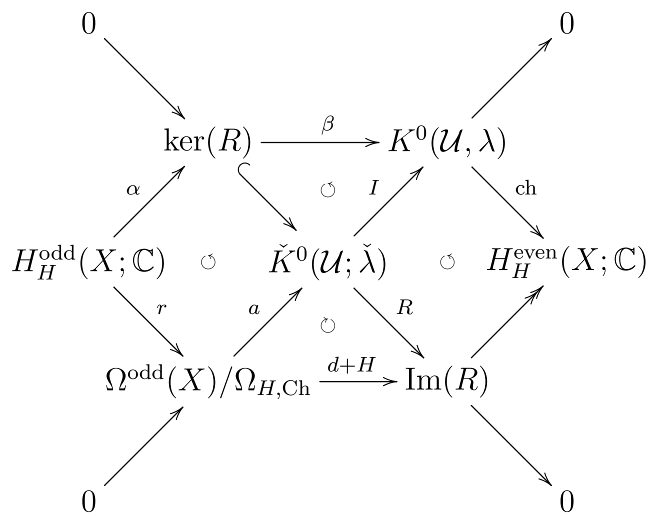
Now observe that $\phi_{\check{\alpha}^\circ}^{-1}$ takes $[(E\oplus G,\Gamma^E\oplus\Gamma^G,\omega-\eta-\zeta)]$ to $[(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},\omega-\eta-\zeta)]$. Also $(F\oplus G,\Gamma^F\oplus\Gamma^G,-\zeta)$ is equivalent to $(E\oplus G,\Gamma^E\oplus\Gamma^G,0)$. $\phi_{\check{\alpha}^\circ}^{-1}$ takes $[(E\oplus G,\Gamma^E\oplus\Gamma^G,0)]$ to $[(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},0)]$.

Accordingly the RHS of (4.5) can be rewritten as

$$\begin{aligned}
&\phi_{\check{\alpha}^\circ}\big([(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},\omega-\eta-\zeta)] \\
&\qquad -[(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},0)]\big) \\
=&\phi_{\check{\alpha}^\circ}\big([(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},\omega-\eta-\zeta)]+[(T,O,0)]-[(T,O,0)] \\
&-[(\phi_{\alpha^\circ}^{-1}(E\oplus G),\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},0)]\big) \\
=&\phi_{\check{\alpha}^\circ}\big([(\phi_{\alpha^\circ}^{-1}(E\oplus G)\oplus T,\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},\omega-\eta-\zeta)] \\
&\qquad -[(\phi_{\alpha^\circ}^{-1}(E\oplus G)\oplus T,\Gamma^E\oplus\Gamma^G-\Pi\cdot\mathbf{1},0)]\big) \\
=&\phi_{\check{\alpha}^\circ}\big([(T,O,\omega-\eta-\zeta)]-[(T,O,0)]\big)=\alpha([\omega-\eta-\zeta]).
\end{aligned}$$

We have to verify that the differential form $\omega-\eta-\zeta$ represents an odd degree twisted cohomology class. Since $[(E,\Gamma^E,\omega)]-[(F,\Gamma^F,\eta)]\in\ker R$, we have $\mathrm{ch}(\Gamma^E)-\mathrm{ch}(\Gamma^F)+(d+H)(\omega-\eta)=0$. Now $(d+H)(\omega-\eta-\zeta)=\mathrm{ch}(\Gamma^E)-\mathrm{ch}(\Gamma^F)-(\mathrm{ch}(\Gamma^E\oplus\Gamma^G)-\mathrm{ch}(\Gamma^F\oplus\Gamma^G))=0$. □

Corollary 4.40. In the following diagram for $\check{K}^0(\mathcal{U};\check{\lambda})$, all square and triangles are commutative and all sequences are exact.



Remark 4.41. When the differential twist is $\check{\lambda} = (\{1\}, \{0\}, \{0\})$, the diagram reduces to the differential K-theory hexagon diagram of Simons and Sullivan (see [35], p. 596).

4.8 Compatibility with change of twist map

We use the same notation $\check{\lambda}$, $\check{\lambda}^\circ$, $\check{\alpha}^\circ$, T and ϕ_α° as in the previous section. Also recall Notation 3.16.

Proposition 4.42. Let $\check{\alpha}' = (\chi'_{ji}, \Pi'_i)$ be an isomorphism $\check{\lambda} \to \check{\lambda}'$ such that $\check{\lambda}' = \check{\lambda} + D\alpha'$. Then the diagram in Corollary 4.40 is natural under change of twist by $\check{\alpha}'$:

(1) $I_{\check{\lambda}'} \circ \phi_{\check{\alpha}'} = \phi_{\alpha'} \circ I_{\check{\lambda}}$.

(2) $\phi_{\check{\alpha}'} \circ a_{\check{\lambda}} = a_{\check{\lambda}'}$.

(3) $R_{\check{\lambda}'} \circ \phi_{\check{\alpha}'} = R_{\check{\lambda}}$.

Proof. (1)

$$\begin{aligned}
&I_{\check{\lambda}'} \circ \phi_{\check{\alpha}'}([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]) \\
=&I_{\check{\lambda}'}([(\phi_{\alpha'} E, \Gamma^E + \Pi' \cdot \mathbf{1}, \omega)] - [(\phi_{\alpha'} F, \Gamma^F + \Pi' \cdot \mathbf{1}, \eta)]) \\
=&[\phi_{\alpha'} E] - [\phi_{\alpha'} F] \\
=&\phi_{\alpha'} \circ I_{\check{\lambda}}([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]).
\end{aligned}$$

(2)

$$\begin{aligned}
\phi_{\check{\alpha}'} \circ a_{\check{\lambda}}(\theta) &= \phi_{\alpha'}([(\phi_{\alpha^\circ} T, \Pi \cdot \mathbf{1}, \theta)] - [(\phi_{\alpha^\circ} T, \Pi \cdot \mathbf{1}, 0)]) \\
&= [(\phi_{\alpha'} \phi_{\alpha^\circ} T, \Pi \cdot \mathbf{1} + \Pi' \cdot \mathbf{1}, \theta)] - [(\phi_{\alpha'} \phi_{\alpha^\circ} T, \Pi \cdot \mathbf{1} + \Pi' \cdot \mathbf{1}, 0)] \\
&= a_{\check{\lambda}''}(\theta).
\end{aligned}$$

(3)

$$\begin{aligned}
&R_{\check{\lambda}'} \circ \phi_{\alpha'}\big([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]\big) \\
&\quad = R_{\check{\lambda}}\big([(\phi_{\alpha'} E, \{\Gamma^E_i + \Pi'_i\}, \omega)] - [(\phi_{\alpha'} F, \{\Gamma^F_i + \Pi'_i\}, \eta)]\big) \\
&\quad = \big(\mathrm{ch}(\Gamma^E + \Pi' \cdot \mathbf{1}) + (d+H)\omega - \mathrm{ch}(\Gamma^F + \Pi' \cdot \mathbf{1}) - (d+H)\eta\big) \\
&\quad = \big(\mathrm{ch}(\Gamma^E) + (d+H)\omega - \mathrm{ch}(\Gamma^F) - (d+H)\eta\big) \qquad \text{by Lemma 3.14.} \\
&\quad = R_{\check{\lambda}}\big([(E, \Gamma^E, \omega)] - [(F, \Gamma^F, \eta)]\big).
\end{aligned}$$

□

Proposition 4.43. Let $\xi \in \Omega^2(X; i\mathbb{R})$ that induces an isomorphism $\check{\lambda} \to \check{\lambda}_\xi$. Then the diagram in Corollary 4.40 is natural under change of twist by ξ:

(1) $\Xi \circ a = a \circ \exp(-\xi)$.

(2) $I \circ \Xi = I$.

(3) $R \circ \Xi = \exp(-\xi) \circ R$.

Appendix A

Finiteness of twisted vector bundles and geometry of $U(1)$-gerbes

This appendix collects a few technical facts which are needed in Section 4.6. It also discusses background material used in Chapter 3. We review a definition of $U(1)$-gerbe with connection and discuss necessary and sufficient conditions for a $U(1)$-gerbe to have a torsion Dixmier-Douady class. After that, we prove that if a $U(1)$-gerbe λ consists of constant maps, then there exists a λ-twisted vector bundle whose transition maps are all constant maps.

A.1 A $U(1)$-gerbe with torsion Dixmier-Douady class

Definition A.1. Let X be a manifold and $\mathcal{U} := \{U_i\}_{i\in\Lambda}$ an open cover of X. A $U(1)$-**gerbe** over X subordinate to $\mathcal{U}$ is a $U(1)$-valued completely normalized Čech 2-cocycle $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$. A **connection** on a $U(1)$-gerbe $\{\lambda_{kji}\}$ on $\mathcal{U}$ is a pair $(\{A_{ji}\}, \{B_i\})$ consisting of a family of differential 1-forms $\{A_{ji} \in \Omega^1(U_{ij}; i\mathbb{R})\}_{i,j\in\Lambda}$ and a family of differential 2-forms $\{B_i \in \Omega^2(U_i; i\mathbb{R})\}_{i\in\Lambda}$ satisfying the following relations:

C1. $\lambda_{kji}\lambda_{lji}^{-1}\lambda_{lki}\lambda_{lkj}^{-1} = 1$

C2. $d\log\lambda_{kji} = A_{ji} + A_{ik} + A_{kj}$

C3. $B_j - B_i = dA_{ji}$

Remark A.2. (1) A $U(1)$-gerbe with connection on $\mathcal{U}$ is therefore a Deligne cocycle of degree 2. Notice that our total differential is $D = d + (-1)^q\delta$ on $\check{C}^p(\mathcal{U}, \Omega^q)$.

(2) From $dB_i = dB_j$ for all $i, j \in \Lambda$, the family of exact 3-forms $\{dB_i\}_{i\in\Lambda}$ defines a global closed differential 3-form H. The differential form H is called the *curvature* of the $U(1)$-gerbe or the *Neveu-Schwarz 3-form.*

(3) Let $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$ be a $U(1)$-gerbe, and $\delta : \check{H}^2(\mathcal{U}, U(1)) \to H^3(X; 2\pi i\mathbb{Z})$ be the connecting map. The image in $H^3_{\mathrm{dR}}(X; i\mathbb{R})$ of the cohomology class $\delta([\lambda]) \in H^3(X; 2\pi i\mathbb{Z})$ coincides with the cohomology class of $H \in H^3_{\mathrm{dR}}(X; i\mathbb{R})$. (See Brylinski [4] p.175, Corollary 4.2.8.)

Recall that a *good cover* is an open cover that every n-fold intersection is contractible for all $n \geq 1$.

Proposition A.3. Let X be a manifold, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ an open cover of X, and $\lambda = \{\lambda_{kji}\}$ a $U(1)$-gerbe on X. If each λ_{kji} is a constant map, then this $U(1)$-gerbe determines a torsion class $\delta([\lambda])$ in $H^3(X; 2\pi i\mathbb{Z})$. Conversely, if a $U(1)$-gerbe λ is defined on a good cover $\mathcal{U}$ and if λ determines a torsion class $\delta([\lambda])$ in $H^3(X; 2\pi i\mathbb{Z})$, then, given any connection $(\{A_{ji}\}, \{B_i\})$ on this $U(1)$-gerbe, there exists a $U(1)$-gerbe with connection $(\{\widetilde{\lambda}_{kji}\}, \{\widetilde{A}_{ji}\}, \{\widetilde{B}_i\})$ that has an underlying $U(1)$-gerbe consisting of a family of constant maps $\widetilde{\lambda}_{kji} : U_{kji} \to U(1)$ such that the difference between $(\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ and $(\{\widetilde{\lambda}_{kji}\}, \{\widetilde{A}_{ji}\}, \{\widetilde{B}_i\})$ is a Deligne coboundary of degree 2.

Proof. We choose any connection $(\{A_{ji}\}, \{B_i\})$ on the given $U(1)$-gerbe λ. Since λ_{kji} are constant maps, it follows that $A_{ji} - A_{ki} + A_{kj} = \lambda_{kji}^{-1} d\lambda_{kji} = 0$. Accordingly if we choose a connection with $A_{ji} \equiv 0$ and $B_i := \zeta|_{U_i}$ for some $\zeta \in \Omega^2(X; i\mathbb{R})$, the triple $(\{\lambda_{kji}\}, \{0\}, \{\zeta|_{U_i}\})$ satisfies the cocycle conditions **C1** to **C3**. Moreover, since the curvature 3-form of this $U(1)$-gerbe with connection is exact, by Remark A.2 (3), it follows that $\delta([\lambda]) \otimes \mathbb{R} = [d\zeta] = 0$, i.e., $\delta([\lambda])$ is a torsion class in $H^3(X, 2\pi i\mathbb{Z})$.

Conversely, suppose a $U(1)$-gerbe λ determines a torsion class $\delta([\lambda])$ in $H^3(X; 2\pi i\mathbb{Z})$. We first choose an arbitrary connection $(\{A_{ji}\}, \{B_i\})$ on the gerbe λ. Using the assumption that the degree 3 cohomology class of the given gerbe is torsion and also the connection $(\{A_{ji}\}, \{B_i\})$, we can obtain another $U(1)$-gerbe with connection $(\{\widetilde{\lambda}_{kji}\}, \{\widetilde{A}_{ji}\}, \{\widetilde{B}_i\})$ that has each $\widetilde{\lambda}_{kji}$ a constant map.

By Remark A.2 (3), $\delta([\lambda])\otimes\mathbb{R} = [H]$, and since $\delta([\lambda])$ is a torsion class, $[H]$ is represented by an exact 2-form on X, i.e., $H = d\zeta$ for some $\zeta \in \Omega^2(X; i\mathbb{R})$. Now from $dB_i = H|_{U_i} = d\zeta|_{U_i}$, we have $d(\zeta|_{U_i} - B_i) = 0$, and since U_i is contractible, by Poincaré's Lemma, $\zeta|_{U_i} - B_i = d\Pi_i$ for some $\Pi_i \in \Omega^1(U_i; i\mathbb{R})$. We define

$$\widetilde{B}_i := B_i + d\Pi_i = \zeta|_{U_i}.$$

Also, from $d(A_{ji} + \Pi_j - \Pi_i) = 0$ on U_{ij}, again by Poincaré's Lemma, there exists $\chi_{ij} \in \Omega^0(U_{ij}; U(1))$ such that

$$(A_{ji} + \Pi_j - \Pi_i) = d\log\chi_{ji}.$$

We define

$$\widetilde{A}_{ji} := A_{ji} + \Pi_j - \Pi_i + d\log\chi_{ji}^{-1}$$

which vanishes, and accordingly define

$$\widetilde{\lambda}_{kji} := \lambda_{kji}\chi_{ji}^{-1}\chi_{ik}^{-1}\chi_{kj}^{-1}.$$

We verify the cocycle condition for $(\{\widetilde{\lambda}_{kji}\}, \{\widetilde{A}_{ji}\}, \{\widetilde{B}_i\})$. On U_{ij}, we have

$\widetilde{B}_j - \widetilde{B}_i = 0 = d\widetilde{A}_{ji}$. This verifies **C3**. Also on U_{ijk},

$$\begin{aligned}(\delta\widetilde{A})_{kji} &= (\delta A)_{kji} - (\delta d\log\chi)_{kji} \\ &= d\log\lambda_{kji} - d\log\chi_{ji} + d\log\chi_{ki} - d\log\chi_{kj} \\ &= d\log\lambda_{kji} + d\log\chi_{ji}^{-1} + d\log\chi_{ik}^{-1} + d\log\chi_{kj}^{-1} = d\log\widetilde{\lambda}_{kji},\end{aligned}$$

so **C2** holds. Condition **C1** follows from $\delta\widetilde{\lambda}_{kji} = \delta(\lambda_{kji}(\delta\chi^{-1})_{kji}) = 1$.

Since $\widetilde{A}_{ji} = 0$ for any i, j, it follows that $d\log\widetilde{\lambda}_{kji} = 0$, and hence each $\widetilde{\lambda}_{kji}$ is a constant map. Notice that the cocycle $(\{\widetilde{\lambda}_{kji}\}, \{\widetilde{A}_{ji}\}, \{\widetilde{B}_i\})$ is obtained by adding to the cocycle $(\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ a degree 2 Deligne coboundary obtained by taking the total exterior derivative of $(\{\chi_{ji}^{-1}\}, \{\Pi_i\}) \in \check{C}^1(\mathcal{U}, \Omega^0) \oplus \check{C}^0(\mathcal{U}, \Omega^1)$. □

A.2 Existence of constant transition maps

Let $1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$ be a short exact sequence of Lie groups, where A is abelian and $f(A)$ belongs to the center of B. There exists a connecting map $\delta_1 : \check{H}^1(X, C) \to \check{H}^2(X, A)$ that extends the induced long exact sequence of pointed sets to $\check{H}^2(X, A)$. This map is defined as follows: Given a class $[c] \in \check{H}^1(X, C)$, take a representative $c = \{c_{ji} : U_{ij} \to C\}$ subordinate to a good cover $\mathcal{U}$ of X that lifts to $b = \{b_{ji} : U_{ij} \to B\}$. We form $(\delta b)_{kji} = b_{ji}b_{ik}b_{kj}$. Since δb is valued in the kernel of g, the inverse image $f^{-1}(\delta b)$ is in $\check{C}^2(\mathcal{U}, A)$, and since δb is a cocycle, so is $f^{-1}(\delta b)$, and hence defines a cohomology class

$[f^{-1}(\delta b)]$. This is the image of $[c]$ under the map δ_1. One can also verify that the map is independent of the choice of representative of $[c]$ and the lifting b. In particular, if $\underline{B}_X$, the sheaf of continuous functions into B, is soft, then by the result of Dixmier and Douady [13] the map δ_1 is a bijection. Both the injectivity and surjectivity proofs use the softness condition.

An observation useful for us is that given $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$ consisting of constant maps, an argument similar to the surjectivity proof of Dixmier and Douady [13] shows that there exists a family of constant transition maps $\{g_{ji} : U_{ij} \to U(n)\}$ such that $g_{kj}g_{ji} = g_{ki}\lambda_{kji}$.

Proposition A.4. Suppose $[a_{kji}] \in \check{H}^2(X, A)$ such that each $a_{kji} : U_{ijk} \to A$ is a constant map. Then there exists a family of constant maps $\{b_{ji} : U_{ij} \to B\}$ satisfying $b_{kj}b_{ji} = b_{ki}a_{kji}$, and hence there exists $[c_{ji}] \in \check{H}^1(X, C)$ with each $c_{ji} : U_{ij} \to C$ is a constant map such that $\delta_1([c_{ji}]) = [a_{kji}]$.

This proposition can be proved in a similar manner to that of Dixmier and Douady's proof of Lemma 22 of [13] (p.279–280). The only difference is that we do not invoke softness and instead have the constant map assumption. We include a proof here for sake of completeness.

Proof of Proposition A.4. Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be a good cover of X. We form the poset $(S, <)$ of pairs (J, b) consisting of a subset J of Λ and a family of constant maps $b = \{b_{ji} : U_{ij} \to B \text{ with } i, j \in J\}$ satisfying $b_{kj}b_{ji} = a_{kji}b_{ki}$ over U_{ijk}. The partial order is defined as follows:

$$(J, b) < (J', b') \text{ if and only if } J \subseteq J' \text{ and } b'_{ij} = b_{ij} \text{ if } i, j \in J.$$

The set S is nonempty and any nonempty chain $(J_\alpha, b_\alpha)_\alpha$ has an upper bound in S, which is $(\widetilde{J} := \cup_\alpha J_\alpha, \widetilde{b} := \cup_\alpha b_\alpha = \{b_{ji} : U_{ij} \to B \text{ where } i, j \in J_\alpha \text{ for some } \alpha\})$. Here b_α denotes $\{b_{ji} : U_{ij} \to B \text{ where } i, j \in J_\alpha\}$. For any fixed α, we have $J_\alpha \subseteq \widetilde{J}$ and $\widetilde{b}_{ij} = b_{ij}$ for any $i, j \in J_\alpha$. Hence by Zorn's Lemma, there exists a maximal element $(\widetilde{J}, \widetilde{b}) \in S$.

We shall now prove $\widetilde{J} = \Lambda$. Suppose $i \in \Lambda - \widetilde{J}$. Then consider the poset $(\hat{S}, <)$ of pairs $(K, \hat{b})$ consisting of a subset K of $\widetilde{J}$ and a family of constant maps $\hat{b} := \{\hat{b}_{ki} : U_{ik} \to B \text{ with } k \in K\}$ satisfying that $b_{kj}\hat{b}_{ji} = a_{kji}\hat{b}_{ki}$ over U_{ijk}. Again, $\hat{S}$ is nonempty and every nonempty chain has an upper bound. By Zorn's Lemma, there exists a maximal element $(\widetilde{K}, \widetilde{\hat{b}}) \in \hat{S}$.

We shall now prove $\widetilde{K} = \widetilde{J}$. Suppose $j \in \widetilde{J} - \widetilde{K}$. Then, for any given $k \in \widetilde{K}$, we can define a unique constant map β_k over U_{ijk} by

$$\beta_k = b_{jk}\hat{b}_{ki}a_{kji}.$$

Then over U_{ijkl},

$$\beta_l = b_{jl}\hat{b}_{li}a_{lji} = b_{jl}(b_{lk}\hat{b}_{ki}a_{lki}^{-1})a_{lji} = b_{jl}(b_{lj}b_{jk}a_{lkj})\hat{b}_{ki}a_{lki}^{-1}a_{lji}$$

$$= b_{jk}\hat{b}_{ki}(a_{lkj}a_{lki}^{-1}a_{lji}) = b_{jk}\hat{b}_{ki}a_{kji} = \beta_k.$$

Hence β is a constant map defined over $U_i \cap U_j \cap (\cup_k U_k)$, and we can extend this map to $U_i \cap U_j$. Since $(\widetilde{K}, \widetilde{\hat{b}}) < (\widetilde{K} \cup \{j\}, \widetilde{\hat{b}} \cup \{\beta\}) \in \hat{S}$, this contradicts to the maximality of $(\widetilde{K}, \widetilde{\hat{b}})$, and thus $\widetilde{K} = \widetilde{J}$.

Next, we set $\hat{b}_{ij} := \hat{b}_{ji}^{-1}$, and take the union of families $(b_{kj})_{j,k\in J}$, $(\hat{b}_{ji})_{j\in J}$, and $(\hat{b}_{ij})_{j\in J}$, and denote it by $\bar{b}$. Since we have $(\widetilde{J}, \widetilde{b}) < (\widetilde{J} \cup \{i\}, \widetilde{b} \cup \bar{b}) \in S$, we get a contradiction to the maximality of $(\widetilde{J}, \widetilde{b})$, and hence $\widetilde{J} = \Lambda$ is proved. Therefore, there exists a family of constant maps $b_{ji} : U_{ij} \to B$ satisfying $b_{kj}b_{ji} = b_{ki}a_{kji}$ for every $i, j, k \in \Lambda$.

Now it follows that there exists a family of constant maps $\{c_{ji} : U_{ij} \to C\}$ defined by $c_{ji} := g(b_{ji})$ satisfying $\delta_1([c_{ji}]) = [a_{kji}]$. $\square$

Appendix B

The odd twisted Chern character

In this appendix, we define the twisted K_1-group with a $U(1)$-gerbe as a topological twist and define the odd twisted Chern character as a map defined on this group. In the course of defining this map, we obtain twisted odd Chern character forms, which were used in Sections 4.6, 4.7, and 4.8.

Notation B.1. Let X be a manifold, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ an open cover on X, and $\lambda = \{\lambda_{kji}\}_{i,j,k\in\Lambda}$ a $U(1)$-gerbe on $\mathcal{U}$ whose Dixmier-Douady class is torsion. We denote by $\mathcal{P}(\mathcal{U}, \lambda)$ a category whose objects are pairs (E, ϕ) consisting of $E \in \mathbf{Bun}(\mathcal{U}, \lambda)$ and $\phi \in \mathrm{Aut}(E)$. A morphism from (E, ϕ) to (E', ϕ') is an isomorphism $\varphi : E \to E'$ such that $\varphi \circ \phi = \phi' \circ \varphi$. We denote the set of isomorphism classes of objects in $\mathcal{P}(\mathcal{U}, \lambda)$ by $\mathtt{Isom}(\mathcal{P}(\mathcal{U}, \lambda))$.

Definition B.2. The **twisted K_1-group** of X defined on an open cover $\mathcal{U}$

with a $U(1)$-gerbe twisting λ is the free abelian group generated by $\mathtt{Isom}(\mathcal{P}(\mathcal{U},\lambda))$ modulo the following relations:

(1) $(E_1 \oplus E_2, \phi_1 \oplus \phi_2) = (E_1, \phi_1) + (E_2, \phi_2)$.

(2) $(E, \phi_1 \circ \phi_2) = (E, \phi_1) + (E, \phi_2)$.

We denote this group by $K_1(\mathcal{U},\lambda)$.

Remark B.3. Let $\lambda = \{\lambda_{kji}\}$ with all $\lambda_{kji} \equiv 1$. The group $K_1(\mathcal{U},\lambda)$ is the *Bass* K_1*-group* associated with the category of (ordinary) complex vector bundles defined on $\mathcal{U}$. Note that this group is in general different from the odd complex K-group $KU^{-1}(X) := [X, U]$, where U is the stablized unitary group. (See Rosenberg [34] p.116.)

Definition B.4. Let X be a manifold, $\mathcal{U}$ a good open cover of X, and $\lambda = \{\lambda_{kji}\}$ a $U(1)$-gerbe on $\mathcal{U}$ whose Dixmier-Douady class is torsion. The **total twisted odd Chern character** is the map

$$\mathrm{Ch} : K_1(X,\lambda) \to H_H^{\mathrm{odd}}(X;\mathbb{C})$$

$$(E,\phi) \mapsto [\mathrm{cs}\left(t \mapsto (1-t)\Gamma^E + t\phi^*\Gamma^E\right)],$$

where Γ^E is a connection on E compatible with $(\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ for some connection $(\{A_{ji}\}, \{B_i\})$ on the $U(1)$-gerbe λ that has the 3-curvature H.

By definition, $\mathrm{cs}\left(t \mapsto (1-t)\Gamma^E + t\phi^*\Gamma^E\right)$ represents an odd twisted cohomology class. We have to verify that $\mathrm{Ch}(E,\phi)$ is independent of choices of connections.

Proposition B.5. (1) $\mathrm{Ch}(E, \phi)$ is independent of the choice of connection on the twisted vector bundle E.

(2) Suppose $\mathcal{U}$ is a good cover. $\mathrm{Ch}(E, \phi)$ is invariant under the change of connections on the $U(1)$-gerbe λ, where the change is of the following form: $\check{\lambda}' = \check{\lambda} + D\check{\alpha}$ for some $\check{\alpha} = (\{1\}, \{\Pi_i\})$, where $\check{\lambda}$ and $\check{\lambda}'$ are the same $U(1)$-gerbes endowed with different connections.

Remark B.6. As in the even case, if $\check{\lambda}' = \check{\lambda}_\xi + D\check{\alpha}$ with nonzero $\xi \in \Omega^2(X; i\mathbb{R})$, then the twisted odd Chern characters obtained by using $\check{\lambda}$ and $\check{\lambda}'$ are related by $\exp(\xi)$.

Proof of Proposition B.5. (1) We fix a $U(1)$-gerbe with connection $\check{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$. Let Γ and ∇ be two different connections on E both associated with $\check{\lambda}$. We observe that

$$\begin{aligned}
\mathrm{Ch}(E, \phi; \nabla) &= [\mathrm{cs}(t \mapsto (1-t)\nabla + t\phi^*\nabla)] \\
&= [\mathrm{cs}(t \mapsto (1-t)\nabla + t\Gamma) + \mathrm{cs}(t \mapsto (1-t)\Gamma + t\phi^*\Gamma) \\
&\qquad + \mathrm{cs}(t \mapsto (1-t)\phi^*\Gamma + t\phi^*\nabla)] \\
&= [\mathrm{cs}(t \mapsto (1-t)\Gamma + t\phi^*\Gamma)] \qquad \text{by Lemma 3.32} \\
&= \mathrm{Ch}(E, \phi; \Gamma).
\end{aligned}$$

(2) Choose Γ on E associated with $\check{\lambda}$. By Proposition B.5 (1), we can

take Γ' on E associated with $\check{\lambda}'$ defined by $\Gamma'_i := \Gamma_i + \Pi_i \cdot \mathbf{1}$ for all $i \in \Lambda$. We see that

$$\begin{aligned} \mathrm{Ch}(E, \phi; \Gamma') &= [\mathrm{cs}(t \mapsto (1-t)\Gamma' + t\phi^*\Gamma')] \\ &= [\mathrm{cs}(t \mapsto (1-t)\Gamma + t\phi^*\Gamma + \Pi \cdot \mathbf{1})] \qquad \text{(B.1)} \\ &= [\mathrm{cs}(t \mapsto (1-t)\Gamma + t\phi^*\Gamma)] = \mathrm{Ch}(E, \phi; \Gamma), \end{aligned}$$

where the third equality follows from a similar calculation appearing in the proof of Proposition 4.19. □

The twisted odd Chern character map satisfies desired properties:

Proposition B.7. (1) If $(E, \phi) = (E', \phi')$, then $\mathrm{Ch}(E, \phi) = \mathrm{Ch}(E', \phi')$.

(2) $\mathrm{Ch}(E_1 \oplus E_2, \phi_1 \oplus \phi_2) = \mathrm{Ch}(E_1, \phi_1) + \mathrm{Ch}(E_2, \phi_2)$.

(3) $\mathrm{Ch}(E, \phi_1 \circ \phi_2) = \mathrm{Ch}(E, \phi_1) + \mathrm{Ch}(E, \phi_2)$.

Proof. (1) Let $\varphi : E \to E'$ be an isomorphism such that $\phi' \circ \varphi = \varphi \circ \phi$. Then

$$\begin{aligned} \mathrm{Ch}(E', \phi') &= \left[\mathrm{cs}\left(t \mapsto (1-t)\Gamma^{E'} + t\phi'^*\Gamma^{E'}\right)\right] \\ &= \left[\mathrm{cs}\left(t \mapsto (1-t)\varphi^*\Gamma^{E'} + t\varphi^*\phi^{*\prime}\Gamma^{E'}\right)\right] \\ &= \left[\mathrm{cs}\left(t \mapsto (1-t)\varphi^*\Gamma^{E'} + t\phi^*(\varphi^*\Gamma^{E'})\right)\right] = \mathrm{Ch}(E, \phi). \end{aligned}$$

(2) Directly follows from additivity of even twisted Chern character.

(3)

$$\begin{aligned}\mathrm{Ch}(E,\phi_1\circ\phi_2) &= [\mathrm{cs}(t\mapsto(1-t)\Gamma^E+t\phi_1^*\phi_2^*\Gamma^E)]\\ &= [\mathrm{cs}(t\mapsto(1-t)\Gamma^E+t\phi_1^*\Gamma^E)]\\ &\quad+[\mathrm{cs}\left(t\mapsto(1-t)\phi_1^*\Gamma^E+t\phi_1^*\phi_2^*\Gamma^E\right)]\\ &= [\mathrm{cs}\left(t\mapsto(1-t)\Gamma^E+t\phi_1^*\Gamma^E\right)]\\ &\quad+[\mathrm{cs}\left(t\mapsto(1-t)\Gamma^E+t\phi_2^*\Gamma^E\right)]\\ &= \mathrm{Ch}(E,\phi_1)+\mathrm{Ch}(E,\phi_2),\end{aligned}$$

where we have used Lemma 3.32 at the third equality. □

The odd twisted Chern character map is functorial.

Proposition B.8. Given a map $f:(Y,\mathcal{V})\to(X,\mathcal{U})$ with $\mathcal{V}=f^{-1}(\mathcal{U})$, the following holds:

$$\mathrm{Ch}(f^*E,(\phi\circ f))=f^*\mathrm{Ch}(E,\phi)$$

Proof. Note that $f^*\phi^*\Gamma^E=(\phi\circ f)^{-1}\circ f^*\Gamma^E\circ(\phi\circ f)+(\phi\circ f)^{-1}d(\phi\circ f)=(\phi\circ f)^*\Gamma^E$. The proof of this statement is similar to the proof of Lemma 4.16. □

The total odd twisted Chern character form respects change of differential twist in a manner that is similar to the even case. (Compare Propositions 3.14 and 3.17.)

Proposition B.9. (1) Let $\check{\lambda}$ and $\check{\lambda}'$ be as in Proposition 3.14. The following holds:

$$\mathrm{Ch}(E', \phi, \Gamma') = \mathrm{Ch}(E, \phi, \Gamma)$$

(2) Let $\xi \in \Omega^2(X; i\mathbb{R})$, and $\check{\lambda}$ and $\check{\lambda}_\xi$ be as in Notation 3.16. The following holds:

$$\mathrm{Ch}(E, \phi, \Gamma_\xi) = \mathrm{Ch}(E, \phi, \Gamma) \wedge \exp(-\xi)$$

Proof. The proof of (1) follows from arguments similar as (B.1). We prove (2). Let $\widetilde{\Gamma}$ be a connection on p^*E defined by pullback of connections on the path $(1-t)\Gamma^E + t\phi^*\Gamma^E$. We have $\mathrm{Ch}(E, \phi, \Gamma^E_\xi) = \mathrm{cs}(t \mapsto (1-t)\Gamma^E_\xi + t\phi^*\Gamma^E_\xi) = \int_I \mathrm{ch}(\widetilde{\Gamma}) \wedge \exp(-p^*\xi) = (\int_I \mathrm{ch}(\widetilde{\Gamma})) \wedge \exp(-\xi) = \mathrm{Ch}(E, \phi, \Gamma^E) \wedge \exp(-\xi)$. $\square$

Appendix C

A smooth variant of Hopkins-Singer differential K-theory

C.1 Introduction

A differential cohomology theory is a construction on smooth manifolds combining topological data and differential form data in a homotopy theoretic way. The first construction of a differential cohomology theory was due to Cheeger and Simons [12] for singular cohomology theory which has applications to geometry. For K-theory, Karoubi [26] developed K-theory with $\mathbb{R}/\mathbb{Z}$ coefficients and Lott [29] developed $\mathbb{R}/\mathbb{Z}$-index theory leading to a construction of differential K-theory and index theorems in differential K-theory. (See [17, 28].) Furthermore, there have been a considerable interest from type IIA/B string theory to represent Ramond-Ramond fields and to formulate

T-duality. (See [15, 25].)

In [24] Hopkins and Singer explicitly constructed a differential cohomology theory for any generalized cohomology theory and hence for K-theory. Their construction provides a correct model of differential K-theory in the sense of the aforementioned homotopy theoretic idea. Following this work, several authors have developed models of differential K-theory by using more geometric cocycle data. (See [8, 10, 17, 20, 22, 28, 35–37].) Furthermore, the Hopkins-Singer model has been revisited by [5, 7] with the idea that differential cohomology theories are ∞-sheaves of spectra. More recently Grady and Sati [21] developed spectral sequences in differential generalized cohomology theories and have opened venues in computational aspects.

One natural question arising at this point is whether all the known models of differential K-theory are isomorphic. Bunke and Schick [9] gave an answer to this question: Any two differential extensions with integration of the same generalized cohomology theory that satisfies certain conditions (such as being rationally even) are uniquely determined up to a unique natural isomorphism. However, it is still interesting to see a direct map between any two different models and proving such a map being an isomorphism has technically intricate aspects.

This appendix is a technical report introducing a smooth variant of the

Hopkins-Singer differential K-theory. This model has an advantage that its cocycles consist of smooth data; continuous maps and singular cochains in the Hopkins-Singer model are replaced by smooth maps and differential forms, respectively. Furthermore, it constitutes an abelian group naturally isomorphic to the original Hopkins-Singer model. Such an aspect facilitates comparisons with other models; we establish a natural isomorphism from the Tradler-Wilson-Zeinalian differential K-theory [36, 37] to the Hopkins-Singer differential K-theory, and hence adding one more item to the following list of known direct comparisons between differential K-theory models.

- Freed-Lott-Klonoff model to Hopkins-Singer model: Klonoff [28] Theorem 4.34 (Even) Freed and Lott [17] Proposition 9.21 (Odd).
- Simons-Sullivan model to Freed-Lott-Klonoff model: Simons and Sullivan [35], Ho [23] Theorem 1. (Even)
- Tradler-Wilson-Zeinalian model to Simons-Sullivan model: Tradler, Wilson, and Zeinalian [37] Remark 3.27. (Even)
- Tradler-Wilson-Zeinalian model to Hekmati-Murray-Schlegel-Vozzo model: Hekmati, Murray, Schlegel, and Vozzo [22] Theorem 4.2. (Odd)
- Ghorokhovsky-Lott model to Freed-Lott-Klonoff model: Ghorokhovsky

and Lott [20] Theorem 1. (Even and odd)

This appendix is organized as follows. Section C.2 outlines definitions and theorems in this appendix. Section C.3 proves Theorem C.1 establishing a natural isomorphism from the smooth variant of the Hopkins-Singer model to the Hopkins-Singer model. Section C.4 proves Theorem C.2 which constructs a natural isomorphism from the Tradler-Wilson-Zeinalian model to the smooth variant of the Hopkins-Singer model. Section C.5 gives a proof of relative de Rham theorem which is used in Section C.3.2.

C.2 Main results

Notation C.1. Throughout this paper, X is a smooth manifold, $\Omega^k(X)$ the differential graded algebra of real-valued differential k-forms on X, $\Omega^k_{\mathrm{cl}}(X)$ (resp. $\Omega^k_{\mathrm{exact}}(X)$) the subalgebra of closed (resp. exact) differential k-forms on X, $C^k(X;\mathbb{R})$ the degree k singular cochain group of X with real coefficients, and $Z^k(X;\mathbb{R})$ the subgroup of $C^k(X;\mathbb{R})$ consisting of degree k cocycles. We denote by I the closed unit interval $[0,1]$, p the projection $p : X \times I \to X$ onto the first factor, and p_i the projection onto the i^{th} factor of the domain. We will also use a notation ψ_t to denote the t-slice maps $\psi_t : X \hookrightarrow X \times I$ defined by $\psi_t(x) = (x,t)$.

Notation C.2. In this paper $\bullet$ is always 0 or 1. We will use the notation $\mathcal{F}_\bullet$ to denote classifying spaces of complex K-theory $\mathcal{F}_0 = BU \times \mathbb{Z}$ and $\mathcal{F}_1 = U$. (We refer readers to Tradler, Wilson, and Zeinalian [37], Section 3 for the models of $BU \times \mathbb{Z}$ and U that we will be using.) We also denote by $c_\bullet$ the sequence of universal Chern character forms $c_{\bullet n} \in \Omega^{2n-\bullet}_{\mathrm{cl}}(\mathcal{F}_\bullet)$ representing universal Chern characters in $H^{2n-\bullet}(\mathcal{F};\mathbb{R})$, defined by $c_{0n} := \mathrm{ch}_n(\nabla_{\mathrm{univ}})$, where ∇_{univ} is the universal connection on the universal bundle $\mathbb{E} \to \mathcal{F}_0$ and $c_{1n} := \Theta_n$ is the n^{th}-universal odd Chern character form on U (This is the degree $(2n-1)$-term in Definition 3.2 of [37]). The space $\mathcal{F}_\bullet$ is endowed with a homotopy commutative H-space structure $m_\bullet : \mathcal{F}_\bullet \times \mathcal{F}_\bullet \to \mathcal{F}_\bullet$. The map m_0 is defined in [37] Definition 3.21 and m_1 in [37] Definition 3.7. We will write $\mathcal{I}_\bullet \in \mathcal{F}_\bullet$ to denote $\mathcal{I}_0 : \mathbb{C}^\infty_{-\infty} \to \mathbb{C}^\infty_{-\infty}$ the orthogonal projection onto $\mathbb{C}^0_{-\infty}$, where $\mathbb{C}^\infty_{-\infty}$ and $\mathbb{C}^0_{-\infty}$ are $\mathbb{C}$-vector spaces spanned by $\{e_i\}_{i\in\mathbb{Z}}$ and $\{e_i : i \in \mathbb{Z}^-\}$, respectively, and $\mathcal{I}_1 := \mathbf{1}$ in U.

Definition C.3. Let $c := c_\bullet$ be a sequence of universal Chern character forms. The **Hopkins-Singer K-theory** of X, denoted by $\widehat{K}^\bullet(X)$, is an abelian group whose elements are equivalence classes of triples (f, h, ω) consisting of the following data.

- A continuous map $f : X \to \mathcal{F}_\bullet$.

- A sequence of $(2n - \bullet)$-forms $\omega = (\omega_n)$, where $\omega_n \in \Omega^{2n-\bullet}(X)$.

- A sequence of $(2n-1-\bullet)$-cochains $h = (h_n)$, where $h_n \in C^{2n-1-\bullet}(X;\mathbb{R})$ satisfying

$$\delta h = f^* \int c - \int \omega. \tag{C.1}$$

Two triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1) are equivalent if and only if the following holds.

- $\omega_0 = \omega_1$, where ω_0 and ω_1 each denotes a sequence of forms.

- There exists an **interpolating triple** $(F, H, p^*\omega_0)$ consisting of a continuous map $F : X \times I \to \mathcal{F}_\bullet$ and a sequence of cochains $H = (H_n)$, where $H_n \in C^{2n-1-\bullet}(X \times I;\mathbb{R})$, satisfying

$$\delta H = F^* \int c - \int p^*\omega_0,$$

  and whose restriction to $X \times \{0\}$ and $X \times \{1\}$ are the triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1), respectively.

The group structure is defined as follows.

$$(f_1, h_1, \omega_1) + (f_2, h_2, \omega_2) := (m_\bullet(f_1, f_2), h_1 + h_2, \omega_1 + \omega_2). \tag{C.2}$$

Lemma C.4. The addition $+$ defined in (C.2) is well-defined and gives $\widehat{K}^\bullet(X)$ the structure of an abelian group.

Proof. Since $m(f_1, f_2)^* \int c = f_1^* \int c + f_2^* \int c$, it is readily seen that the RHS of (C.2) satisfies the triple relation. Suppose $(F, H, p^*\omega_1)$ is an interpolating triple between (f_1, h_1, ω_1) and (f_1', h_1', ω_1). The triple $(m_\bullet(F, p^*f_2), H + p^*h_2, p^*\omega_1 + p^*\omega_2)$ interpolates between $(m_\bullet(f_1, f_2), h_1 + h_2, \omega_1 + \omega_2)$ and $(m_\bullet(f_1', f_2), h_1' + h_2, \omega_1 + \omega_2)$.

The operation + being an abelian group operation follows from Lemmas 3.9, 3.23, and 3.24 (2) of [37] and verifying it needs several lemmas we shall prove in the following section. We shall give a proof in Section C.3.4. □

Instead of singular cochains and continuous maps in Definition C.3, the following definition uses differential forms and smooth maps.

Definition C.5. Let $c := c_\bullet$ be a sequence of universal Chern character forms. The **smooth Hopkins-Singer differential K-theory** of X, denoted by $\check{K}^\bullet(X)$, is an abelian group whose elements are equivalence classes of triples (f, h, ω) consisting of the following data.

- A smooth map $f : X \to \mathcal{F}_\bullet$.
- A sequence of $2n$-forms $\omega = (\omega_n)$, where $\omega_n \in \Omega^{2n-\bullet}(X)$.
- A sequence of $(2n - 1 - \bullet)$-forms $h = (h_n)$, where $h_n \in \Omega^{2n-1-\bullet}(X)$, satisfying

$$dh = f^*c - \omega. \tag{C.3}$$

Two triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1) are equivalent if and only if the following holds.

- $\omega_0 = \omega_1$, where ω_0 and ω_1 each denotes a sequence of forms.
- There exists an **interpolating triple** $(F, H, p^*\omega_0)$, consisting of a smooth map $F : X \times I \to \mathcal{F}_\bullet$, and a sequence of differential forms $H = (H_n)$, where $H_n \in \Omega^{2n-1-\bullet}(X \times I)$, satisfying

  $$dH = F^*c - p^*\omega_0,$$

  and whose restriction to $X \times \{0\}$ and $X \times \{1\}$ are the triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1), respectively.

The group structure is defined as follows.

$$(f_1, h_1, \omega_1) + (f_2, h_2, \omega_2) := (m_\bullet(f_1, f_2), h_1 + h_2, \omega_1 + \omega_2). \tag{C.4}$$

Lemma C.6. The addition + defined in (C.4) is well-defined and gives $\check{K}^\bullet(X)$ the structure of an abelian group.

Proof. The RHS of (C.4) satisfying the triple axiom follows from

$$m(f_1, f_2)^*c = f_1^*c + f_2^*c,$$

and the well-definedness is verified by the same argument as in the proof of

Lemma C.4. We give a proof that the operation + is commutative in Section C.3.4. □

Theorem C.1. Let $c := c_\bullet$ be a sequence of universal Chern character forms. The assignment

$$\begin{aligned} \check{K}^\bullet(X) &\to \widehat{K}^\bullet(X) \\ [(f,h,\omega)] &\mapsto \left[(f, \int h, \omega)\right] \end{aligned} \tag{C.5}$$

is an isomorphism of abelian groups that is natural in X.

Proof. See Section C.3. □

Definition C.7. Let $c := c_\bullet$ be a sequence of universal Chern character forms and $f_0, f_1 : X \to \mathcal{F}_\bullet$ homotopic smooth maps via a smooth homotopy $F : X \times I \to \mathcal{F}_\bullet$. The **Chern-Simons form** of F is

$$\mathrm{cs}(F) := (-1)^{\bullet - 1} \int_I \mathrm{ch}(F), \tag{C.6}$$

where $\mathrm{ch}(F) := F^* c$.

Definition C.8. Two smooth maps $f_0, f_1 : X \to \mathcal{F}$ are **cs-equivalent** and denoted by $f_0 \sim_{\mathrm{cs}} f_1$ if there exists a smooth homotopy $F : X \times I \to \mathcal{F}$ between f_0 and f_1 such that $\mathrm{cs}(F) \in \Omega^*_{\mathrm{exact}}(X)$.

Proposition C.9. $\sim_{\mathrm{cs}}$ is an equivalence relation.

Proof. $f \sim_{\mathrm{cs}} f$ follows from $\mathrm{cs}(f \circ p) = 0$. Suppose $f_0 \sim_{\mathrm{cs}} f_1$ with homotopy F between f_0 and f_1 such that $\mathrm{cs}(F)$ is exact. Define $G(x,t) := F(x, 1-t)$. Then $\mathrm{cs}(G) = (-1)^{\bullet-1}\int_I \mathrm{ch}(G) = -(-1)^{\bullet-1}\int_I \mathrm{ch}(F) = -\mathrm{cs}(F)$, which is exact. Hence $f_1 \sim_{\mathrm{cs}} f_0$. Finally, suppose $f_0 \sim_{\mathrm{cs}} f_1$ through homotopy F, and $f_1 \sim_{\mathrm{cs}} f_2$ through homotopy G. Define

$$H(x,t) := \begin{cases} F(x, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} .$$

Then H is a homotopy interpolating f_0 and f_2, and

$$\mathrm{cs}(H) = (-1)^{\bullet-1}\int_I \mathrm{ch}(H) = (-1)^{\bullet-1}\int_I \mathrm{ch}(F) + (-1)^{\bullet-1}\int_I \mathrm{ch}(G)$$

is an element of $\Omega^*_{\mathrm{exact}}(X)$. Hence $f_0 \sim_{\mathrm{cs}} f_2$. □

Notation C.10. We denote by $[f]_{\mathrm{cs}}$ the cs-equivalence class of a map $f : X \to \mathcal{F}_\bullet$.

Definition C.11. The Tradler-Wilson-Zeinalian differential K-theory of X is the set

$$\widehat{K}^\bullet_{TWZ}(X) := \{[f]_{\mathrm{cs}} : f : X \to \mathcal{F} \text{ is a smooth map.}\}$$

endowed with a structure of abelian group induced by $m_\bullet$.

Theorem C.2. The assignment

$$\Phi : \widehat{K}^{\bullet}_{TWZ}(X) \to \check{K}^{\bullet}(X)$$

$$[f]_{\text{cs}} \mapsto [(f, 0, \text{ch}(f))]$$

is an isomorphism of abelian groups that is natural in X.

Proof. See Section C.4. □

C.3 Proof of Theorem C.1

C.3.1 Well-definedness

Suppose two triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1) are equivalent; i.e., $\omega_0 = \omega_1$ and there exists an interpolating triple $(F, H, p^*\omega)$ satisfying $dH = F^*c - p^*\omega$. Integrating both sides we obtain: $\delta \int H = F^* \int c - \int p^*\omega$, and hence $(F, \int H, p^*\omega)$ is a triple that interpolates between $(f_0, \int h_0, \omega_0)$ and $(f_1, \int h_1, \omega_1)$.

C.3.2 Injectivity

For any two triples (f_0, h_0, ω_0) and (f_1, h_1, ω_1), we assume that $(f_0, \int h_0, \omega_0)$ and $(f_1, \int h_1, \omega_1)$ are equivalent, i.e., $\omega_0 = \omega_1$, and there exists an interpolating triple $(F, H, p^*\omega)$ with the triple relation

$$\delta H = F^* \int c - \int p^*\omega. \tag{C.7}$$

We pause our proof and prove a lemma that is necessary in proving injectivity. Recall that, given f_0, f_1, and F be as in the above paragraph, there exists a homotopy $G : X \times I \times I \to \mathcal{F}_\bullet$ between F and a smooth approximation $\overline{F}$ of F, which fixes the ends: $\overline{F}(x,0) = F(x,0) = f_0(x)$ and $\overline{F}(x,1) = F(x,1) = f_1(x)$. By a smooth approximation $\overline{F}$ we mean a smooth map $\overline{F}$ that is homotopic to a continuous map F by a continuous homotopy G.

Lemma C.12. $F^* \int c - \overline{F}^* \int c = \delta K$ for some K, where K is defined in the proof.

Proof. Let G be as in the above paragraph. For $G^* \int c \in C^*(X \times I^2)$, we take the slant product with I and then take the exterior derivative. By the derivation formula for the slant product, we have:

$$\begin{aligned}\delta\left(G^* \int c \Big/ I\right) &= \left(\delta G^* \int c\right) \Big/ I + (-1)^{\left|G^* \int c\right| + |I|} G^* \int c \Big/ \partial I \\ &= (-1)^\bullet \left(F^* \int c - \overline{F}^* \int c\right).\end{aligned}$$

We denote $(-1)^\bullet G^* \int c \Big/ I$ by K. Note that $K|_{M \times \{0,1\}} = 0$. □

Now we resume our proof. By Lemma C.12, equation (C.7) can be written as

$$\delta(H - K) = \overline{F}^* \int c - \int p^* \omega_0.$$

Now consider a sequence of differential forms

$$\overline{F}^* c - p^* \omega_0 - d(\mathfrak{H}) \tag{C.8}$$

where $\mathfrak{H} := (1-t)h_0 + th_1$. This is a sequence of closed forms on $X \times I$, that vanishes on $X \times \{0,1\}$. Hence $\overline{F}^* c - p^* \omega_0 - d(\mathfrak{H}) \in \Omega^*_{\mathrm{cl}}(X \times I, X \times \{0,1\})$. If we integrate (C.8), we obtain

$$\int \overline{F}^* c - \int p^* \omega_0 - \int d(\mathfrak{H}) = \delta(H - K) - \delta \int \mathfrak{H}.$$

By the relative de Rham theorem (see Appendix A), it follows that $\overline{F}^* c - p^* \omega_0 - d(\mathfrak{H}) = d'\xi$, for some $\xi \in \Omega^{*-1}(X \times I)$ where $\xi \equiv 0$ on $X \times \{0,1\}$. (Here d' is the differential of the relative complex — see Appendix A.) We thus have an equality $\overline{F}^* c - p^* \omega_0 = d(\xi + \mathfrak{H})$ in $X \times I$, which is the triple relation for $(\overline{F}, \xi + \mathfrak{H}, p^* \omega_0)$. This triple interpolates between (f_0, h_0, ω_0) and (f_1, h_1, ω_1).

C.3.3 Surjectivity

We consider the special case that the classifying map $f : X \to \mathcal{F}_\bullet$ is given by a smooth map and then the general case.

Case I: We choose an element $[(f, h, \omega)] \in \widehat{K}^\bullet(X)$, with the property that there exists a representative (f, h, ω) with f smooth. Consider the triple relation for the representative (f, h, ω)

$$\delta h = f^* \int c - \int \omega.$$

By de Rham theorem, $f^*c - \omega = d\xi$ for some sequence of differential forms ξ on X, and hence $\delta h = \delta \int \xi$. Since $h - \int \xi$ represents a cohomology class, there exists $\eta \in \Omega^*_{\mathrm{cl}}(X)$ such that $[h - \int \xi] = [\int \eta]$ or equivalently, $h - \int \xi = \int \eta + \delta\mu$ for some cochain μ. We write $\zeta := \xi + \eta$

Now we consider the triple (f, ζ, ω). The map (C.5) takes this triple to $(f, \int \zeta, \omega)$. We claim that $(f, \int \zeta, \omega)$ and $(f, \int \zeta + \delta\mu, \omega)$ are equivalent. The following lemma is standard but we give a proof for sake of completeness.

Lemma C.13. If α, $\beta \in Z^n(X; \mathbb{R})$ such that $\alpha - \beta = \delta k$ for some $k \in C^{n-1}(X; \mathbb{R})$, then there exists $L \in Z^n(X \times I; \mathbb{R})$ such that $\psi_0^* L = \alpha$ and $\psi_1^* L = \beta$. (Recall Notation C.1.)

Proof. Consider an interval I as a CW complex with two 0-cells a, b, and a 1-cell e. We define cochains $e^* \in C^1(I; \mathbb{R})$ by $e \mapsto 1$, $a^* \in C^0(I; \mathbb{R})$ by $a \mapsto 1$ and $b \mapsto 0$, and $b^* \in C^0(I; \mathbb{R})$ by $a \mapsto 0$ and $b \mapsto 1$. Then, for any 0-cell v in X,

$$(\psi_0^* p_2^*) a^*(v) = a^*(p_{2*}\psi_{0*} v) = a^*(a) = 1,$$

and similarly,

$$(\psi_0^* p_2^*) b^*(v) = b^*(a) = 0$$

$$(\psi_1^* p_2^*) a^*(v) = a^*(b) = 0$$

$$(\psi_1^* p_2^*) b^*(v) = b^*(b) = 1.$$

Also, for any 1-cell σ in X,

$$(\psi_0^* p_2^*) e^*(\sigma) = e^*(p_{2*}\psi_{0*}(\sigma)) = e^*(0) = 0,$$

and similarly, $(\psi_1^* p_2^*) e^*(\sigma) = 0$. We also have:

$$\delta a^*(e) = a^* \partial(e) = a^*(b - a) = -1 \quad \Leftrightarrow \quad \delta a^* = -e^*$$

$$\delta b^*(e) = b^* \partial(e) = b^*(b - a) = 1 \quad \Leftrightarrow \quad \delta b^* = e^*.$$

Since there is no 2-cell in I, we also have $\delta e^* = 0$.

Now we take $L := p_1^* \alpha \cup p_2^* a^* + p_1^* \beta \cup p_2^* b^* + (-1)^n p_1^* k \cup p_2^* e^*$. We see that:

$$\delta L = (-1)^n p_1^* \alpha \cup \delta p_2^* a^* + (-1)^n p_1^* \beta \cup \delta p_2^* b^* + (-1)^n \delta p_1^* k \cup p_2^* e^* + 0$$

$$= (-1)^n \left(-p_1^* \alpha \cup p_2^* e^* + p_1^* \beta \cup p_2^* e^* + \delta p_1^* k \cup p_2^* e^* \right) = 0, \text{since } \delta k = \alpha - \beta,$$

and also:

$$\psi_0^* L = (\psi_0^* p_1^*) \alpha \cup (\psi_0^* p_2^*) a^* + (\psi_0^* p_1^*) \beta \cup (\psi_0^* p_2^*) b^* + (-1)^n (\psi_0^* p_1^*) k \cup (\psi_0^* p_2^*) e^*$$

$$= \alpha \cup 1 + \beta \cup 0 + 0 = \alpha,$$

$$\psi_1^* L = (\psi_1^* p_1^*) \alpha \cup (\psi_1^* p_2^*) a^* + (\psi_1^* p_1^*) \beta \cup (\psi_1^* p_2^*) b^* + (-1)^n (\psi_1^* p_1^*) k \cup (\psi_1^* p_2^*) e^*$$

$$= \alpha \cup 0 + \beta \cup 1 + 0 = \beta.$$

□

We continue proof of surjectivity. By Lemma C.13, there exists a cocycle L on $X \times I$ such that $\psi_0^* L = 0$ and $\psi_1^* L = \delta\mu$. Using this L, we form the triple $(f \circ p, p^* \int \zeta + L, p^*\omega)$. This triple restricts to $(f, \int \zeta, \omega)$ and $(f, \int \zeta + \delta\mu, \omega)$ at each end. Furthermore,

$$\delta\left(p^* \int \zeta + L\right) = p^*\delta \int \xi = p^*\left(f^* \int c - \int \omega\right) = (f \circ p)^* \int c - \int p^*\omega,$$

where in the first equality, we used the fact that η is a closed form. Hence the claim. Therefore, for any $[(f, h, \omega)] \in \widehat{K}^\bullet(X)$ with smooth f, there is an element $[(f, \zeta, \omega)] \in \check{K}^\bullet(X)$ in the preimage.

Case II: We consider a triple whose classifying map is not necessarily smooth. Given any $[(f, h, \omega)] \in \widehat{K}^\bullet(X)$, it suffices to show that there exists a triple $(\overline{f}, h', \omega)$, with a smooth classifying map $\overline{f}$, equivalent to (f, h, ω). We consider a smooth approximation $\overline{f}$ of f satisfying that $\overline{f}$ and f are homotopic through a homotopy g with $\overline{f} = g(-, 1)$.

Lemma C.14. Let (f_0, h, ω) be a triple representing an element of $\widehat{K}^\bullet(X)$. Suppose f_0 is homotopic to f_1 via a homotopy F. Then the triple

$$\left(f_1, h + (-1)^{|c|+1} F^* \int c \Big/ I, \omega\right)$$

is equivalent to (f_0, h, ω).

Proof. We take an interpolating triple

$$\left(F, p^*h + (-1)^{|c|+1} G^* \int c \Big/ I, p^*\omega\right),$$

where G is a homotopy between F and $f_0 \circ p$ defined by

$$\begin{aligned} G : X \times I \times I &\to \mathcal{F}_\bullet \\ (x,t,s) \mapsto G(x,t,s) &:= \begin{cases} F(x,t) & \text{if } t \le s \\ F(x,s) & \text{if } t \ge s \end{cases}. \end{aligned} \tag{C.9}$$

In particular, $G(x,t,0) = F(x,0) = f_0 \circ p$ and $G(x,t,1) = F(x,t) = F$.

Let $\widetilde{\psi_s} : X \times I \to X \times I \times I$ be a s-slice map defined by $(x,t) \mapsto (x,t,s)$.

Since

$$\begin{aligned} \widetilde{\psi}_0^* \left(G^* \int c \Big/ I\right) &= (G \circ \widetilde{\psi_0})^* \int c \Big/ I = (f_0 \circ p)^* \int c \Big/ I = 0 \\ \widetilde{\psi}_1^* \left(G^* \int c \Big/ I\right) &= (G \circ \widetilde{\psi_1})^* \int c \Big/ I = F^* \int c \Big/ I, \end{aligned}$$

the triple $\left(F, p^*h + (-1)^{|c|+1} G^* \int c \Big/ I, p^*\omega\right)$ interpolates the between given two triples. We verify the triple relation:

$$\begin{aligned}
&\delta\left(p^*h+(-1)^{|c|+1}G^*\int c\Big/I\right)=p^*\delta h+\delta\left((-1)^{|c|+1}G^*\int c\Big/I\right)\\
=&p^*(f_0^*\int c-\int\omega)+(-1)^{|c|+1}\delta G^*\int c\Big/I+G^*\int c\Big/\partial I\\
=&p^*f_0^*\int c-\int p^*\omega+\left(\widetilde{\psi_1}^*G^*\int c-\widetilde{\psi_0}^*G^*\int c\right)\\
=&p^*f_0^*\int c-\int p^*\omega+\left(F^*\int c-(f_0\circ p)^*\int c\right)\\
=&F^*\int c-\int p^*\omega.
\end{aligned}$$

□

Therefore, by Lemma C.14, the triple $\left(\overline{f}, h+(-1)^{|c|+1}F^*\int c\Big/I,\omega\right)$ is equivalent to (f,h,ω). Now a preimage of $[(\overline{f},h',\omega)]$ can be found, by Case I.

C.3.4 Group homomorphism and naturality

We first prove that (C.2) and (C.4) is an abelian group operation as claimed in Lemmas C.4 and C.6, respectively. Note that $(\mathcal{I}_\bullet,0,0)$ is the identity. The existence of inverses will follow from similar arguments. We prove associativity presently.

Consider any three triples (f_1,h_1,ω_1), (f_2,h_2,ω_2), and (f_3,h_3,ω_3) in $\widehat{K}^\bullet(X)$. By the argument in Case II in Section C.3.3, we may assume that

f_1, f_2, and f_3 are smooth. Consider the following triples

$$\begin{aligned}&(m_\bullet(f_1, m_\bullet(f_2, f_3)), h_1 + h_2 + h_3, \omega_1 + \omega_2 + \omega_3)\\ &(m_\bullet(m_\bullet(f_1, f_2), f_3)), h_1 + h_2 + h_3, \omega_1 + \omega_2 + \omega_3)\end{aligned} \tag{C.10}$$

By Lemmas 3.24 (2) and 3.9 of [37] for m_0 and m_1, respectively, two maps $m_\bullet(f_1, m_\bullet(f_2, f_3))$ and $m_\bullet(m_\bullet(f_1, f_2), f_3)$ are cs-equivalent such that $\mathrm{cs}(\Gamma_\bullet) = 0$ for some homotopy $\Gamma_\bullet$ between $m_\bullet(f_1, m_\bullet(f_2, f_3))$ and $m_\bullet(m_\bullet(f_1, f_2), f_3)$. From Lemmas C.14 (using the homotopy $\Gamma_\bullet$) and C.13, it follows that the triples in (C.10) are equivalent.

Now suppose any three triples (f_1, h_1, ω_1), (f_2, h_2, ω_2), and (f_3, h_3, ω_3) are in $\check{K}^\bullet(X)$. Again $m_\bullet(f_1, m_\bullet(f_2, f_3)) \sim_{\mathrm{cs}} m_\bullet(m_\bullet(f_1, f_2), f_3)$ by the same reason, and triples again of the form (C.10) are equivalent by a similar argument in Section C.4.1 below and the fact that $\mathrm{cs}(\Gamma_\bullet) = 0$.

It is readily seen that the map (C.5) is a group homomorphism. It is natural in X by the change of variables formula.

C.4 Proof of Theorem C.2

C.4.1 Well-definedness

Suppose $f_0 \sim_{\mathrm{cs}} f_1$ through a homotopy F. We have to show that two triples $(f_0, 0, \mathrm{ch}(f_0))$ and $(f_1, 0, \mathrm{ch}(f_1))$ in $\check{K}^\bullet(X)$ are equivalent. Since $\mathrm{cs}(F)$ is exact, it follows that $\mathrm{ch}(f_0) = \mathrm{ch}(f_1)$. We define an interpolating triple by

$(F, \mathrm{cs}(G), p^*\mathrm{ch}(f_0))$, where G is a homotopy between $f_0 \circ p$ and F defined in (C.9). We have the triple relation

$$d\mathrm{cs}(G) = \mathrm{ch}(F) - \mathrm{ch}(f_0 \circ p),$$

and the triple $(F, \mathrm{cs}(G), p^*\mathrm{ch}(f_0))$ becomes $(f_0, \mathrm{cs}(f_0 \circ p \circ \widetilde{\psi}_0), \mathrm{ch}(f_0))$ (resp. $(f_1, \mathrm{cs}(F), \mathrm{ch}(f_0)))$ when it is restricted to $X \times \{0\}$ (resp. $X \times \{1\}$). We claim that triples $(f_1, \mathrm{cs}(F), \mathrm{ch}(f_0))$ and $(f_1, 0, \mathrm{ch}(f_0))$ are equivalent. This can be easily verified by applying the following Lemma.

Lemma C.15. If α, $\beta \in \Omega^n_{\mathrm{cl}}(X; \mathbb{R})$ are such that $\alpha - \beta = d\gamma$ for some $\gamma \in \Omega^{n-1}(X; \mathbb{R})$, then there exists $\xi \in \Omega^n_{\mathrm{cl}}(X \times I; \mathbb{R})$ such that $\psi_0^*\xi = \alpha$ and $\psi_1^*\xi = \beta$.

Proof. Set $\xi := (1-t)p^*\alpha + tp^*\beta - dt \wedge p^*\gamma \in \Omega^n(X \times I; \mathbb{R})$. Then $d\xi = -dt \wedge p^*\alpha + dt \wedge p^*\beta + dt \wedge p^*d\gamma = 0$, $\psi_0^*\xi = \alpha$, and $\psi_1^*\xi = \beta$. □

Since $\mathrm{cs}(F)$ is exact, we may write $\mathrm{cs}(F) := d\mu$. We apply Lemma C.15 with $\alpha = d\mu$ and $\beta = 0$. More explicitly, the interpolating triple between $(f_1, \mathrm{cs}(F), \mathrm{ch}(f_0))$ and $(f_1, 0, \mathrm{ch}(f_0))$ is $(f_1 \circ p, \xi, p^*\mathrm{ch}(f_1))$ where

$$\xi := (1-t)p^*d\mu - dt \wedge p^*\mu.$$

We see that $d\xi = -dt \wedge p^*d\mu + dt \wedge p^*d\mu = 0$, $\psi_0^*\xi = d\mu$, and $\psi_1^*\xi = 0$. The triple relation is easily verified: $d\xi = 0 = \mathrm{ch}(f_1 \circ p) - p^*\mathrm{ch}(f_1)$. Thus the map is well-defined.

C.4.2 Injectivity

Suppose two triples $(f_0, 0, \mathrm{ch}(f_0))$ and $(f_1, 0, \mathrm{ch}(f_1))$ in $\check{K}^\bullet(X)$ are equivalent. i.e. $\mathrm{ch}(f_0) = \mathrm{ch}(f_1)$ and there exists a homotopy F between f_0 and f_1, such that

$$dH = \mathrm{ch}(F) - p^*\mathrm{ch}(f_0)$$

for some sequence of differential forms $H = (H_n)$ where $H_n \in \Omega^{2n-1-\bullet}(X \times I; \mathbb{R})$. Integrating both sides along I, we get

$$\begin{aligned}\int_I \mathrm{ch}(F) - 0 &= \int_I \mathrm{ch}(F) - \int_I p^*\mathrm{ch}(f_0)\\ &= \int_I dH + 0 = \int_I dH + (-1)^{|H|-1}\int_{\partial I} H = d\int_I H.\end{aligned}$$

This shows that $\mathrm{cs}(F)$ is exact, and hence $f_0 \sim_{\mathrm{cs}} f_1$.

C.4.3 Surjectivity

We need two lemmas. The following lemma follows from [37].

Lemma C.16 (Strong Venice Lemma)**.** Given a sequence of differential forms $h = (h_n)$ with $h_n \in \Omega^{2n-1-\bullet}(X)$ and a smooth map $f_1 : X \to \mathcal{F}_\bullet$, there exists a smooth map $f_0 : X \to \mathcal{F}_\bullet$ and a smooth homotopy $F : X \times I \to \mathcal{F}_\bullet$ between f_0 and f_1 such that $\mathrm{cs}(F) = h$.

Proof. Given such a sequence h, Theorem 3.17 (2) of [37] shows that there exists a homotopy $G : X \times I \to \mathcal{F}_\bullet$ such that $G(x,1) = \mathcal{I}_\bullet$ and $\mathrm{cs}(G) = h$. (See Notation C.2 for the definition of $\mathcal{I}_\bullet$.)

Accordingly, define a homotopy $F : X \times I \to \mathcal{F}_\bullet$ by

$$F(x,t) := m_\bullet(G(x,t), p(f_1(x))).$$

Note that $F(x,1) = m_\bullet(G(x,1), p(f_1(x))) = f_1(x)$ and

$$\mathrm{cs}(F) = (-1)^{\bullet-1}\int_I \mathrm{ch}(G) + (-1)^{\bullet-1}\int_I p\circ f_1 = \mathrm{cs}(G) + 0 = h.$$

At $t=0$, $F(x,0) = m_\bullet(G(x,0), p(f_1(x)))$ which we denote by $f_0(x)$. □

Lemma C.17. Let (f_0, h, ω) be a triple representing an element of $\check{K}^\bullet(X)$. Suppose f_0 is homotopic to f_1 via a homotopy F. Then the triple $(f_1, h + \mathrm{cs}(F), \omega)$ is equivalent to (f_0, h, ω).

Proof. We choose $(F, p^*h + \mathrm{cs}(G), p^*\omega)$, where G is a homotopy between F and $f_0 \circ p$ defined in (C.9). We verify that this triple interpolates between (f_0, h, ω) and $(f_1, h + \mathrm{cs}(F), \omega)$. First, we have

$$\widetilde{\psi}_0^*(F, p^*h + \mathrm{cs}(G), p^*\omega) = (f_0, h + \mathrm{cs}(G\circ\widetilde{\psi}_0), \widetilde{\psi}_0^* p^*\omega) = (f_0, h, \omega),$$

$$\widetilde{\psi}_1^*(F, p^*h + \mathrm{cs}(G), p^*\omega) = (f_1, h + \mathrm{cs}(G\circ\widetilde{\psi}_1), \widetilde{\psi}_1^* p^*\omega) = (f_1, h + \mathrm{cs}(F), \omega),$$

where $\widetilde{\psi}_s$ is as defined in the proof of Lemma C.14.

The triple condition then follows from

$$d\left(p^*h + \mathrm{cs}(G)\right) = p^* f_0^* c - p^*\omega + F^*c - p^* f_0^* c = F^*c - p^*\omega.$$

□

We now prove surjectivity. Take any representative (f_1, h, ω) of any element in $\check{K}^\bullet(X)$. Applying Lemma C.16 with h and f_1, we may write $h = \mathrm{cs}(F)$ where F is a homotopy between f_0 and f_1 for some f_0. Note that $\mathrm{ch}(f_0) = \omega$, because

$$\mathrm{ch}(f_1) - \mathrm{ch}(f_0) = dh = \mathrm{ch}(f_1) - \omega.$$

By Lemma C.17, the triple $(f_0, 0, \omega)$ is equivalent to the triple $(f_1, \mathrm{cs}(F), \omega)$. Since $(f_0, 0, \omega)$ is a representative of the image of $[f_0] \in \widehat{K}^0_{\mathrm{TWZ}}(X)$, $[f_0]$ is a preimage of $(f_1, h, \omega) \in \check{K}^0(X)$.

C.4.4 Group homomorphism and naturality

The given map is a group homomorphism since $\mathrm{ch}(m(f_0, f_1)) = \mathrm{ch}(f_0) + \mathrm{ch}(f_1)$. It is natural in X by the naturality of ch.

C.5 Relative de Rham theorem

We state and prove a variant of de Rham theorem for relative cohomology groups. This result is certainly well-known and classical, but we did not find a reference. Throughout this appendix, let X be a smooth manifold, Y a closed submanifold, and $\imath : Y \hookrightarrow X$ a smooth embedding.

Definition C.18. The **relative de Rham complex** $(\Omega^\bullet(X,Y), d'_\bullet)$ is defined by $\Omega^k(X,Y) := \ker \imath_k^*$ and $d'_k := d|_{\ker \imath_k^*}$ for each $k \geq 0$, where $\imath_k^* : \Omega^k(X) \to \Omega^k(Y)$ is the restriction map.

Since $\imath^*$ commutes with d, the image of d'_k is contained in $\Omega^{k+1}(X,Y)$, hence the pair $(\Omega^\bullet(X,Y), d'_\bullet)$ is a complex. Also note that the kernel of d'_k is $\Omega^k_{\rm cl}(X) \cap \ker i_k^*$.

Definition C.19. The degree k **relative de Rham cohomology groups** of the pair (X,Y) is defined by

$$H^k_{\rm dR}(X,Y) := \ker d'_k \big/ \mathrm{Im}\ d'_{k-1}, \quad k \in \mathbb{Z}.$$

Lemma C.20. (1) In the following diagram, the rows are exact, and the squares are commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(X,Y) & \xrightarrow{\hookrightarrow} & \Omega^k(X) & \xrightarrow{\imath_k^*} & \Omega^k(Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'_k} & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \\ 0 & \longrightarrow & \Omega^{k+1}(X,Y) & \xrightarrow{\hookrightarrow} & \Omega^{k+1}(X) & \xrightarrow{\imath_{k+1}^*} & \Omega^{k+1}(Y) & \longrightarrow & 0 \end{array}$$

(2) The following sequence of cohomology groups is exact.

$$0 \to H^0_{\mathrm{dR}}(X,Y) \to H^0_{\mathrm{dR}}(X) \to H^0_{\mathrm{dR}}(Y) \xrightarrow{\delta_{\mathrm{dR}}} H^1_{\mathrm{dR}}(X,Y) \to$$

$$H^1_{\mathrm{dR}}(X) \to H^1_{\mathrm{dR}}(Y) \xrightarrow{\delta_{\mathrm{dR}}} \ldots .$$

Proof. (1) The surjectivity of the map $\imath_k^* : \Omega^k(Y) \to \Omega^k(X)$ for $k \in \mathbb{Z}$ follows from Whitney's embedding theorem. The diagram is commutative because pull-back commutes with d. (2) follows from the snake lemma. □

Proposition C.21 (Relative de Rham Theorem)**.** The assignment

$$\int : H^k_{\mathrm{dR}}(X,Y) \to H^k(X,Y;\mathbb{R})$$

$$[\omega] \mapsto [\int \omega],$$

is a natural isomorphism of groups for each $k \in \mathbb{Z}$.

Proof. We use the 5-lemma. Consider the following diagram.

$$\begin{array}{ccccccccc} H^{k-1}_{\mathrm{dR}}(X) & \xrightarrow{\imath_{k-1}^*} & H^{k-1}_{\mathrm{dR}}(Y) & \xrightarrow{\delta_{\mathrm{dR}}} & H^k_{\mathrm{dR}}(X,Y) & \longrightarrow & H^k_{\mathrm{dR}}(X) & \xrightarrow{\imath_k^*} & H^k_{\mathrm{dR}}(Y) \\ \cong\downarrow{\scriptstyle \int} & & \cong\downarrow{\scriptstyle \int} & & \downarrow & & \cong\downarrow{\scriptstyle \int} & & \cong\downarrow{\scriptstyle \int} \\ H^k(X) & \xrightarrow{\imath_{k-1}^*} & H^k(Y) & \xrightarrow{\delta} & H^k(X,Y) & \longrightarrow & H^k(X) & \xrightarrow{\imath_k^*} & H^k(Y) \end{array}$$

The first and the last squares are commutative by naturality of de Rham theorem. The third square commutes by Stokes' formula. We verify the commutativity of the second square. Take any $[\theta] \in H^{k-1}_{\mathrm{dR}}(Y)$. There exists $\eta \in \Omega^{k-1}(X)$ whose restriction to Y is θ. The restriction of the differential form $d\eta \in \Omega^k(X)$ to Y is identically zero. Hence $d\eta$ is a representative of the cohomology class $\delta_{\mathrm{dR}}([\theta]) \in H^k_{\mathrm{dR}}(X, Y)$. Applying the vertical map, we obtain $[\int d\eta]$. Now we apply the vertical map to $[\theta]$ and then the connecting map. This gives $\delta[\int \theta] \in H^k(X, Y; \mathbb{R})$. Since $\int \eta$ is a cochain in X that restricts to $\int \theta$ in Y, the k-cochain $\int d\eta$ represents the singular relative cohomology class $\delta[\int \theta]$, since $\imath_k^*(\int d\eta)$ is vanishing. $\square$

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