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Interstructure Lattices and Types of Peano Arithmetic

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INTERSTRUCTURE LATTICES AND TYPES OF PEANO ARITHMETIC

by

ATHAR ABDUL-QUADER

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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Doctor of Philosophy.

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Abstract

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ATHAR ABDUL-QUADER

Advisor: Professor Roman Kossak

The collection of elementary substructures of a model of PA forms a lattice, and is referred to as the substructure lattice of the model. In this thesis, we study substructure and interstructure lattices of models of PA. We apply techniques used in studying these lattices to other problems in the model theory of PA.

In Chapter 2, we study a problem that had its origin in Simpson ([Sim74]), who used arithmetic forcing to show that every countable model of PA has an expansion to PA^* that is pointwise definable. Enayat ([Ena88]) later showed that there are 2^{\aleph_0} models with the property that every expansion to PA^* is pointwise definable. In this Chapter, we use techniques involved in representations of lattices to show that there is a model of PA with this property which contains an infinite descending chain of elementary cuts.

In Chapter 3, we study the question of when subsets can be coded in elementary end extensions with prescribed interstructure lattices. This problem originated in Gaifman [Gai76], who showed that every model of PA has a conservative, minimal elementary end extension. That is, every model of PA has a minimal elementary end extension which codes only definable sets. Kossak

and Paris [KP92] showed that if a model is countable and a subset X can be coded in any elementary end extension, then it can be coded in a minimal extension. Schmerl ([Sch14] and [Sch15]) extended this work by considering which collections of sets can be the sets coded in a minimal elementary end extension. In this Chapter, we extend this work to other lattices. We study two questions: given a countable model \mathcal{M} , which sets can be coded in an elementary end extension such that the interstructure lattice is some prescribed finite distributive lattice; and, given an arbitrary model \mathcal{M} , which sets can be coded in an elementary end extension whose interstructure lattice is a finite Boolean algebra?

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In 1972 Dr. Mohammed Abdul-Quader left India for a new life with around \$8 in his pocket. He worked harder than anyone I ever knew, raising his entire family of 9 sisters and 2 brothers after his father met his demise. When I decided to leave my job in order to attend graduate school, he took me back into his home with no questions asked. On March 19, 2017, he returned to his Lord. Anything I could say about him in this space would be inadequate, so let me close with this: Dad, I miss you, I love you, and I dedicate this work to your memory. May your soul be at peace with the Creator of Peace.

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Chapter 1

Introduction

Peano Arithmetic, abbreviated PA, arose from an attempt to formalize number theory. This process of formalizing mathematics, in the broader context of first order logic, has led to deeper insights in the foundations of mathematics. As part of mathematical logic, the model theory of PA is deeply related to other foundational topics in mathematics, including set theory and reverse mathematics. It is also connected to other areas of mathematics, including number theory, combinatorics, and algebra. In addition, much of the work relevant to this thesis arises from the general study of lattice theory and the representation theory of lattices in particular.

1.1 Preliminaries

\mathcal{L}_{PA} is the first-order language of PA, consisting of the constant symbols 0 and 1, the relation symbol \leq , and the binary function symbols $+$ and \times . The axioms for PA include the theory of the non-negative parts of discretely ordered rings, as well as the induction schema: for each formula ϕ in the language of arithmetic, $\forall \bar{b} [(\phi(0, \bar{b}) \wedge \forall x(\phi(x, \bar{b}) \rightarrow \phi(x + 1, \bar{b}))) \rightarrow \forall x\phi(x, \bar{b})]$. The full list of axioms can be found in any standard text (for example, [Kay91]).

A model of arithmetic is a tuple $\mathcal{M} = (M, 0, 1, \leq, +, \times)$ satisfying the axioms for PA. We use script letters $\mathcal{M}, \mathcal{N}, \dots$ to refer to models of PA, and we use the corresponding Roman letters M, N, \dots to denote their respective universes. The standard model is the set of natural numbers, \mathbb{N} , with its usual interpretations of \leq , $+$, and \times . This model is of course not the only model of PA: by the compactness theorem of first order logic, there must be non-standard models, which contain elements greater than any natural number.

Let \mathcal{L}' be a language extending \mathcal{L}_{PA} . An *expansion* of a model \mathcal{M} of PA to \mathcal{L}' is a model $\mathcal{M}^* = (M, 0, 1, \leq, +, \times, \dots)$ with the same universe M as \mathcal{M} , and the same interpretations for all the symbols of \mathcal{L}_{PA} . We are particularly interested in expansions upon adding a single unary predicate, whose interpretation will be some set $X \subseteq M$. For a fixed language $\mathcal{L}' \supseteq \mathcal{L}_{PA}$, PA^* is the axioms of PA together with the induction schema for all formulas in the expanded language.

A set $X \subseteq M$ is *definable* if there is a formula $\phi(x, \bar{y})$ in the language of

arithmetic and a tuple $\bar{b} \in M$ such that $X = \{x \in M : \mathcal{M} \models \phi(x, \bar{b})\}$ (this is the usual model-theoretic definition of a definable set). The collection of all definable sets in \mathcal{M} is denoted $\text{Def}(\mathcal{M})$. A set X is called *inductive* if the expansion (\mathcal{M}, X) satisfies PA^* in the language with a unary predicate for X . That is, for every formula $\phi(x, \bar{y})$ in the expanded language with a unary predicate X , and every tuple $\bar{b} \in M$, we have $(\mathcal{M}, X) \models (\phi(0, \bar{b}) \wedge \forall x(\phi(x, \bar{b}) \rightarrow \phi(x+1, \bar{b}))) \rightarrow \forall x\phi(x, \bar{b})$. The collection of all inductive sets in \mathcal{M} is denoted $\text{Ind}(\mathcal{M})$. A set X is called a *class* of \mathcal{M} if for all $a \in M$, the set $\{x \in X : x \leq a\} \in \text{Def}(\mathcal{M})$. The set of all classes of a model \mathcal{M} is denoted $\text{Class}(\mathcal{M})$.

Every definable set is inductive, and every inductive set is a class. For countable models these inclusions are proper ([KS06, Chapter 1]). It is also known that there are uncountable models where these are all equal. A model with the property that every class is definable is called *rather classless*.

Let $\mathcal{M} \subseteq \mathcal{N}$ be models of PA. We say \mathcal{N} is a *cofinal* extension of \mathcal{M} if for every $a \in N$, there is $b \in M$ such that $\mathcal{N} \models a < b$. We denote this $\mathcal{M} \subseteq_{\text{cof}} \mathcal{N}$. We say \mathcal{N} is an *end* extension of \mathcal{M} , denoted $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, if for every $a \in M, b \in N \setminus M, \mathcal{N} \models a < b$. If these extensions are elementary, we write $\mathcal{M} \preceq_{\text{cof}} \mathcal{N}$ or $\mathcal{M} \preceq_{\text{end}} \mathcal{N}$, respectively. In fact, Gaifman's Splitting Theorem ([Gai72]) says that if \mathcal{M} and \mathcal{N} are models of PA such that $\mathcal{M} \subseteq_{\text{cof}} \mathcal{N}$, then $\mathcal{M} \preceq_{\text{cof}} \mathcal{N}$.

If $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$ are models of PA, and $X \subseteq M$, we say that X is *coded* in \mathcal{N} if there is $Y \in \text{Def}(\mathcal{N})$ such that $X = Y \cap M$. The collection of all subsets of M

coded in \mathcal{N} is denoted $\text{Cod}(\mathcal{N}/\mathcal{M})$. In the case of an elementary end extension, it is always the case that $\text{Def}(\mathcal{M}) \subseteq \text{Cod}(\mathcal{N}/\mathcal{M})$. If $\text{Cod}(\mathcal{N}/\mathcal{M}) = \text{Def}(\mathcal{M})$, the extension is called *conservative*, denoted $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$. Every conservative extension is necessarily an end extension. For every $\mathcal{M} \models \text{PA}$, $\mathbb{N} \subseteq_{\text{end}} \mathcal{M}$, and we denote $\text{Cod}(\mathcal{M}/\mathbb{N})$ as $\text{SSy}(\mathcal{M})$, called the *standard system* of \mathcal{M} . If $X \subseteq M$ is coded in an end extension \mathcal{N} , then $X \in \text{Class}(\mathcal{M})$. If a model is countable, it has classes that are not coded in any end extension ([KP92]).

The standard system of any model of PA is a *Scott set*. $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ is a Scott set if $(\omega, \mathfrak{X}) \models \text{WKL}_0$. That is, \mathfrak{X} is a Boolean Algebra of sets, is closed under relative Turing computability, and satisfies Weak König's Lemma (WKL): if $T \in \mathfrak{X}$ is a set of codes of nodes of an infinite binary tree, then there is a set $B \in \mathfrak{X}$ coding an infinite branch through T . A classic problem in the study of PA asks whether every Scott set is the standard system of some model of PA.

Formulas with no set quantifiers and only bounded first-order quantifiers are said to be Σ_0^0 . If ϕ is of the form $\forall x\psi(x)$, where ψ is Σ_n^0 , then ϕ is Π_{n+1}^0 . If ϕ is $\exists x\psi(x)$ where ψ is Π_n^0 , then ϕ is Σ_{n+1}^0 . We identify definable sets X with the (smallest) complexity of a formula defining them. If a set can be defined by both a Σ_n^0 and a Π_n^0 formula, we say that the set is Δ_n^0 .

Given a model $\mathcal{M} \models \text{PA}$ and a collection $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, we can consider second order properties of the structures $(\mathcal{M}, \mathfrak{X})$. $\text{I}\Sigma_n$ is the induction scheme for Σ_n^0 formulas. $\Delta_1^0 - \text{CA}$ is the Δ_1^0 comprehension axiom scheme. RCA_0^* is $\Delta_1^0 - \text{CA} + \text{I}\Sigma_0$, and WKL_0^* is $\text{RCA}_0^* + \text{WKL}$.

Adopting terminology from [Sch14], we say that $\mathfrak{X}_0 \subseteq \mathfrak{X}$ *generates* \mathfrak{X} if,

whenever $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \mathfrak{X}$ and $(\mathcal{M}, \mathfrak{X}_1) \models \Delta_1^0 - \text{CA}$, then $\mathfrak{X}_1 = \mathfrak{X}$. In other words, \mathfrak{X} is the closure of \mathfrak{X}_0 under Δ_1^0 -definability. \mathfrak{X} is *countably generated* if it is generated by a countable subset. For example, if \mathcal{M} is any model of PA, then $\text{Def}(\mathcal{M})$ is countably generated, even if $\text{Def}(\mathcal{M})$ is not itself countable, as the set of 0-definable sets is countable and generates $\text{Def}(\mathcal{M})$. To see this, let $X \in \text{Def}(\mathcal{M})$ be the set $\{x : \mathcal{M} \models \phi(x, b)\}$ for some $b \in M$. Then the set $Y = \{\langle x, y \rangle : \mathcal{M} \models \phi(x, y)\}$ is 0-definable, and $X = \{x : \langle x, b \rangle \in Y\}$ is Δ_1^0 -definable from Y .

If $\mathcal{M} \prec \mathcal{N}$ are such that whenever $\mathcal{M} \preceq \mathcal{K} \preceq \mathcal{N}$, then $\mathcal{K} = \mathcal{M}$ or $\mathcal{K} = \mathcal{N}$, then \mathcal{N} is called a *minimal* extension of \mathcal{M} . If we also have that $\mathcal{K} \preceq \mathcal{N}$ implies that either $\mathcal{K} \preceq \mathcal{M}$ or $\mathcal{K} = \mathcal{N}$, then \mathcal{N} is called a *superminimal* extension of \mathcal{M} .

A classic result in the model theory of PA is the theorem of MacDowell and Specker [MDS61], stating that every model \mathcal{M} of PA has an elementary end extension. In fact, their proof shows that every model \mathcal{M} has a conservative elementary end extension. This feature turned out to be an important part of the result as there are some models whose elementary end extensions are always conservative. The MacDowell-Specker Theorem has been refined many times, notably by Gaifman [Gai76], who showed that every model of PA has a minimal conservative elementary end extension. This result by Gaifman initiated a systematic study of the relationship between the interstructure lattice of an extension and the sets which can be coded in that extension.

Schmerl [Sch14] expanded this work, in some sense generalizing MacDowell-

Specker. Schmerl showed that if $\mathcal{M} \models \text{PA}$, $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ is countably generated, contains $\text{Def}(\mathcal{M})$, and $(\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0^*$, then there is a finitely generated elementary end extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. In particular, if $\mathfrak{X} = \text{Def}(\mathcal{M})$, then this result gives another proof of the MacDowell-Specker Theorem: every model of PA has a conservative elementary end extension.

In the same paper, Schmerl showed that if, in addition, \mathcal{M} and \mathfrak{X} are countable, then there is a minimal elementary end extension such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. This generalized a result in [KP92] which studied whether a single set which could be coded in some elementary end extension can be coded in a minimal end extension.

Simpson [Sim74] showed that every countable model $\mathcal{M} \models \text{PA}$ has a pointwise definable expansion $(\mathcal{M}, X) \models \text{PA}^*$, where $X \subseteq M$. A structure \mathcal{M} is pointwise definable if for each $a \in M$, the set $\{a\}$ is definable without parameters. Simpson's argument uses arithmetic forcing, which produces an inductive generic $G \subseteq M$. We review arithmetic forcing here. If $\mathcal{M} \models \text{PA}$ is countable, let $\mathbb{P} = 2^{<M}$, and if $p, q \in \mathbb{P}$, then $p \leq q$ if and only if $p \supseteq q$. A set $D \subseteq \mathbb{P}$ is called *dense* if for each $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq p$. $F \subseteq \mathbb{P}$ is a *filter* if it is non-empty, for each $p, q \in F$ there is $r \in F$ such that $r \leq p$ and $r \leq q$, and for each $p \in F$ and $q \in \mathbb{P}$, if $p \leq q$ then $q \in F$. A set $X \subseteq \mathbb{P}$ is *generic* if it is a filter and, for each definable dense subset D , $X \cap D \neq \emptyset$. We can identify a generic G with a set $X \subseteq M$ by letting X be the set of all $x \in M$ such that there is some $p \in G$ with $p(x) = 1$. More details on arithmetic forcing can be found in Chapter 6 of [KS06] which shows the basic results, including that

every generic G is undefinable and inductive.

One may ask whether arithmetic forcing can be used to find an undefinable, inductive generic $G \subseteq M$ so that no new elements are definable in (\mathcal{M}, G) . Enayat [Ena88] showed that this is impossible: there are 2^{\aleph_0} non-isomorphic models \mathcal{M} with the property that for any undefinable class $X \subseteq M$, the expansion (\mathcal{M}, X) is pointwise definable. Enayat's result inspires the following definition:

Definition 1. Let $\mathcal{M} \models \text{PA}$. If, for every undefinable class X of M , (\mathcal{M}, X) is pointwise definable, then \mathcal{M} is called an *Enayat model*.

Clearly, every prime model is Enayat. [Ena88] showed that, for each completion T of PA , any superminimal conservative extension of the prime model of T is Enayat. By a similar proof, if α is a countable ordinal, then the union of an elementary chain of superminimal conservative extensions of length α is Enayat.

The work in this thesis is based in large part on the discussion of substructure lattices of models of PA given in [KS06, Chapter 4]. We will repeat some definitions and review some history here.

A *lattice* is a partial order (L, \leq) such that every pair of elements of L has a least upper bound (\vee) and a greatest lower bound (\wedge). A lattice is called *distributive* if the \vee and \wedge operations obey the distributive law: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Given $\mathcal{M} \models \text{PA}$, the set of all $\mathcal{K} \prec \mathcal{M}$ forms a lattice under inclusion, called the *substructure lattice* of \mathcal{M} and denoted $\text{Lt}(\mathcal{M})$. Given

$\mathcal{M} \prec \mathcal{N}$, the *interstructure lattice*, denoted $\text{Lt}(\mathcal{N}/\mathcal{M})$ is the set of all \mathcal{K} such that $\mathcal{M} \preceq \mathcal{K} \preceq \mathcal{N}$.

Given $1 \leq n \in \omega$, the lattice $\mathbf{n} = (\{0, \dots, n-1\}, <)$. A lattice of this form is referred to as a finite chain. The lattice $\mathbf{B}_n = (\mathcal{P}(\{0, \dots, n-1\}), \subseteq)$ is the Boolean algebra on an n -element set.

Many results on substructure and interstructure lattices are known. First, it is known that every substructure lattice must be \aleph_1 -algebraic. Given a lattice L , $a \in L$ is *compact* if whenever $X \subseteq L$ and $a \leq \bigvee X$, then there is a finite $Y \subseteq X$ such that $a \leq \bigvee Y$. L is *algebraic* if each $a \in L$ is a supremum of a set of compact elements. If κ is a cardinal, then L is κ -algebraic if it is algebraic and each $a \in L$ has less than κ compact predecessors.

The first positive result on substructure and interstructure lattices came from Gaifman, as mentioned before, who showed in [Gai76] that every model of PA has a minimal elementary end extension. In the same paper, Gaifman showed that if D is a finite distributive lattice, then every countable model $\mathcal{M} \models \text{PA}$ has an elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$. The most general result on distributive lattices is due to George Mills, in [Mil79], who showed that if \mathcal{M} is any model of PA and D is any \aleph_1 -algebraic distributive lattice, then there is $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$.

Regarding non-distributive lattices, the first results were due to Paris ([Par77]) and Wilkie ([Wil77]). Paris showed that the lattice \mathbf{M}_3 can be embedded in a substructure lattice: this is the lattice with a bottom element, $\mathbf{0}$, a top element $\mathbf{1}$, and three incomparable elements in between. Wilkie showed

that the pentagon lattice \mathbf{N}_5 can be realized as a substructure lattice: this is the lattice containing $\mathbf{0}, \mathbf{1}$, and three elements a, b , and c such that $\mathbf{0} < a < \mathbf{1}$ and $\mathbf{0} < b < c < \mathbf{1}$. In fact, his proof can be used to show that every countable model \mathcal{M} has an elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$. Later, Schmerl ([Sch86]) showed that \mathbf{M}_3 can be realized as a substructure lattice. Still, there is no complete picture as to which lattices can be realized as substructure lattices: for example, it is open whether the lattice \mathbf{M}_{16} (containing a top, bottom, and 16 incomparable elements in between) can be realized as a substructure lattice.

Given a lattice L , a *rank function* $r : L \rightarrow L$ is a function satisfying the following for all $x, y \in L$:

1. $x \leq r(x)$
2. $r(r(x)) = r(x)$
3. $r(x) \leq r(y)$ or $r(y) \leq r(x)$, and
4. $r(x \vee y) = r(x) \vee r(y)$

We refer to (L, \leq, r) as a *ranked lattice*. The *rankset* of a ranked lattice is the set $R = \{r(x) : x \in L\}$. Interstructure (and therefore substructure) lattices can be realized as ranked lattices in the following way: given $\mathcal{M} \prec \mathcal{N}$, define $r : \text{Lt}(\mathcal{N}/\mathcal{M}) \rightarrow \text{Lt}(\mathcal{N}/\mathcal{M})$ by letting $r(\mathcal{K}) = \{a \in N : \exists y \in K \mathcal{N} \models a \leq y\}$. That is, $r(\mathcal{K})$ is the closure of \mathcal{K} under initial segments in \mathcal{N} . By Gaifman's

splitting theorem [Gai72], for each $\mathcal{K} \prec \mathcal{N}$, there is a unique \mathcal{K}' such that $\mathcal{K} \preceq_{\text{cof}} \mathcal{K}' \preceq_{\text{end}} \mathcal{N}$: the rank of \mathcal{K} , then, is this \mathcal{K}' .

Given a set A , $\text{Eq}(A)$ is the set of equivalence relations on A . This again forms a lattice under inclusion, with top element $\mathbf{1}_A$ being the trivial equivalence relation A^2 , and bottom element $\mathbf{0}_A$ being the discrete relation $\{(x, x) : x \in A\}$. Given a finite lattice L , a *representation* of L is an injection $\alpha : L \rightarrow \text{Eq}(A)$ for some set A , such that $\alpha(0) = \mathbf{1}_A$, $\alpha(1) = \mathbf{0}_A$, and $\alpha(x \vee y) = \alpha(x) \wedge \alpha(y)$ for all $x, y \in L$. If $\alpha : L \rightarrow \text{Eq}(A)$ is a representation and $B \subseteq A$, then $\alpha|_B : L \rightarrow \text{Eq}(B)$ is the function defined by $(\alpha|_B)(r) = \alpha(r) \cap B^2$.

Definition 2. If $\alpha : L \rightarrow \text{Eq}(A)$ and $\beta : L \rightarrow \text{Eq}(B)$ are representations, we say $\alpha \cong \beta$ if there is a bijection $f : A \rightarrow B$ such that for all $r \in L$ and $x, y \in A$, $(x, y) \in \alpha(r) \iff (f(x), f(y)) \in \beta(r)$. In this case, we say f *confirms* the isomorphism.

The main results of the thesis are in Chapters 2 and 3. Chapter 2 discusses Enayat models and shows that, for each completion T of PA , there are continuum-many non-isomorphic Enayat models whose substructure lattices contain an infinite descending chain.

In Chapter 3, we study the lattice problem in connection with the work done in [Sch14]. That is, we study questions of the form: given a lattice L , a (possibly countable) model $\mathcal{M} \models \text{PA}$ and a collection $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, if $(\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0^*$ and \mathfrak{X} is countably generated and contains $\text{Def}(\mathcal{M})$, is there $\mathcal{N} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$?

Chapter 2

Enayat Models

2.1 Introduction

Simpson [Sim74] showed that every countable model $\mathcal{M} \models \text{PA}$ has an expansion $(\mathcal{M}, X) \models \text{PA}^*$ that is pointwise definable. This is done by arithmetic forcing. The natural question is whether, in general, one can use arithmetic forcing to obtain expansions of a model in which the definable elements coincide with those of the underlying model. Enayat [Ena88] showed that this is impossible by proving that there is $\mathcal{M} \models \text{PA}$ such that for each undefinable class X of \mathcal{M} , the expansion (\mathcal{M}, X) is pointwise definable. Here we examine this result and look for other models with this property.

2.2 Enayat Models

The ultimate goal of this project is to give a complete characterization of Enayat models in terms of better-known model-theoretic properties. Currently, there are few properties known about Enayat models. First we show that Enayat models cannot have proper cofinal submodels.

Lemma 1. *Let $\mathcal{M} \models \text{PA}$ be countable and suppose $\mathcal{K} \prec_{\text{cof}} \mathcal{M}$ is a proper submodel. Then \mathcal{M} is not Enayat.*

Proof. Because \mathcal{K} is countable, we can find an undefinable inductive subset X of \mathcal{K} by arithmetic forcing ([Fef64]). We can extend this X uniquely to $Y \subseteq M$: for each $a \in K$ we there is some formula $\phi_a(x)$ (possibly using parameters from K) which defines $\{x \in K : x \leq a \wedge x \in X\}$. Then let $Y = \bigcup_{a \in K} \{x \in M : \mathcal{M} \models \phi_a(x)\}$ and one can show that $(\mathcal{K}, X) \prec (\mathcal{M}, Y)$. This result is due independently to Kotlarski and Schmerl; see [KS06, Theorem 1.3.7]. Since $\text{Scl}^{(\mathcal{M}, Y)}(0) \subseteq \mathcal{K}$, \mathcal{M} is not Enayat. \square

Lemma 1 gives us an easy characterization of which finite lattices can appear as the substructure lattices of an Enayat model. Given two lattices L_1 and L_2 , the lattice $L = L_1 \oplus L_2$ is the lattice formed by identifying the top element of L_1 with the bottom element of L_2 . In particular, for any lattice L , $L \oplus \mathbf{2}$ is the lattice formed by adding one new element above the top element of L . As an example, if \mathcal{N} is a superminimal elementary extension of \mathcal{M} , then $\text{Lt}(\mathcal{N}) \cong \text{Lt}(\mathcal{M}) \oplus \mathbf{2}$.

Corollary 1.

- (1) Let $\mathcal{M} \models \text{PA}$ be an Enayat model. If $\text{Lt}(\mathcal{M})$ is finite, then it is of the form $L \oplus \mathbf{2}$ where L is some finite lattice.
- (2) Let L be a finite lattice, \mathbb{T} a completion of PA and $\mathbb{T} \neq \text{TA}$. If there is $\mathcal{N} \models \mathbb{T}$ such that $\text{Lt}(\mathcal{N}) = L$, then there is an Enayat $\mathcal{M} \models \mathbb{T}$ such that $\text{Lt}(\mathcal{M}) \cong L \oplus \mathbf{2}$.

Proof. To prove (1), all we need to show here is that the top element of $\text{Lt}(\mathcal{M})$ cannot have more than one immediate predecessor. Suppose there are two: \mathcal{K}_1 and \mathcal{K}_2 . Notice that, since these are immediate predecessors of \mathcal{M} , the extensions $\mathcal{K}_i \prec \mathcal{M}$ are minimal. By Gaifman's Splitting Theorem ([Gai72]), there is $\bar{\mathcal{K}}_i$ such that $\mathcal{K}_i \preceq_{\text{cof}} \bar{\mathcal{K}}_i \preceq_{\text{end}} \mathcal{M}$. By minimality, for each i , either $\mathcal{K}_i = \bar{\mathcal{K}}_i$ or $\bar{\mathcal{K}}_i = \mathcal{M}$. So either $\mathcal{K}_i \prec_{\text{end}} \mathcal{M}$ or $\mathcal{K}_i \prec_{\text{cof}} \mathcal{M}$. Suppose neither \mathcal{K}_1 nor \mathcal{K}_2 is cofinal in \mathcal{M} , and therefore they are both cuts. Because \mathcal{K}_1 and \mathcal{K}_2 are incomparable in $\text{Lt}(\mathcal{M})$, there are $a \in K_1 \setminus K_2$ and $b \in K_2 \setminus K_1$. Then either $\mathcal{M} \models a < b$ or $\mathcal{M} \models b < a$. Because the \mathcal{K}_i are cuts, in the former case, that means $a \in K_2$ and in the latter case, $b \in K_1$. These are both contradictions, so one of the \mathcal{K}_i must be a cofinal submodel of \mathcal{M} . This is impossible if \mathcal{M} is Enayat.

For the proof of (2), let $\mathcal{M}_T \models \mathbb{T}$ be a prime model of T . Since there is $\mathcal{N} \models \mathbb{T}$ with $\text{Lt}(\mathcal{N}) = L$, then by Theorems 4.5.21 and 4.5.22 in [KS06], there is a cofinal extension \mathcal{K} of \mathcal{M}_T such that $\text{Lt}(\mathcal{K}) = L$. Let \mathcal{M} be a superminimal conservative extension of \mathcal{K} . Theorem 2.2.13 in [KS06] shows

that this \mathcal{M} must be Enayat. \square

The following remains open:

Question 1. *Which finite lattices can be realized as the substructure lattice of an Enayat model of TA?*

We can modify the above proof to get that, for a finite lattice L , if there is a model $\mathcal{M} \models \text{TA}$ such that $\text{Lt}(\mathcal{M}) = \mathbf{2} \oplus L$, then there is an Enayat model of TA whose substructure lattice is $\mathbf{2} \oplus L \oplus \mathbf{2}$. Other Enayat models of TA can be found using results in the next section. As an example, there is an Enayat model of TA whose substructure lattice is isomorphic to $\mathbf{B}_2 \oplus \mathbf{2}$, showing that substructure lattices of models of TA need not be isomorphic to a lattice of the form $\mathbf{2} \oplus L \oplus \mathbf{2}$ for some finite lattice L .

Corollary 1 implies that there are Enayat models of PA whose substructure lattice is isomorphic to $\mathbf{N}_5 \oplus \mathbf{2}$. It is unknown whether there is an Enayat model of TA whose substructure lattice is isomorphic to this lattice.

2.3 Enayat Models with Infinite Descending Sequences of Elementary Submodels

In this section, we first show some examples of Enayat models. Then we prove our main result, which is that there is an Enayat model whose substructure lattice forms an infinite descending chain.

Proposition 1. *Suppose \mathcal{M} is a conservative extension of each of its elementary cuts and that all elementary cuts are well-ordered by inclusion. Then if \mathcal{N} is a superminimal conservative extension of \mathcal{M} , it is Enayat.*

Before we prove this, we note that we can find many examples of Enayat models in this way. Corollary 2.2.12 of [KS06] states that every countable model of PA has a superminimal conservative extension. We can then form countable elementary chains of superminimal conservative extensions, which, by Proposition 1, would be Enayat models. That is, if α is a countable ordinal, $\mathcal{N} = \bigcup_{\beta < \alpha} \mathcal{M}_\beta$, where \mathcal{M}_0 is prime, $\mathcal{M}_{\beta+1}$ is a superminimal conservative extension of \mathcal{M}_β , and $\mathcal{M}_\lambda = \bigcup_{\beta < \lambda} \mathcal{M}_\beta$ whenever λ is a limit ordinal, then \mathcal{N} is Enayat.

As another example, suppose \mathcal{M}_0 is a nonstandard prime model and \mathcal{N}_0 is a cofinal extension of \mathcal{M}_0 . Let \mathcal{M}_1 be a superminimal conservative extension of \mathcal{N}_0 . We can find a cofinal extension \mathcal{N}_1 of \mathcal{M}_1 such that $\mathcal{N}_0 \prec_{\text{cons}} \mathcal{N}_1$. Then, by Proposition 1, if \mathcal{N} is a superminimal conservative extension of \mathcal{N}_1 it is Enayat. This shows that we can obtain Enayat models which are not chains of superminimal conservative extensions.

Proof. The proof will begin with a lemma that is very similar to [KS06, Theorem 2.2.13].

Lemma 2. *Let $\mathcal{N} \models \text{PA}$, X an undefinable class of \mathcal{N} , and $a \in N$. Let $\mathcal{M}_a = \text{sup}^N(\text{Scl}(a))$. If $\mathcal{M}_a \prec_{\text{cons}} \mathcal{N}$, then there is $b > \mathcal{M}_a$ such that $b \in \text{dcl}^{(N, X)}(a)$.*

Proof. Since $\mathcal{M}_a \prec_{\text{cons}} \mathcal{N}$, the set $X \cap \mathcal{M}_a$ is definable in \mathcal{M}_a by some formula

$\phi(x, c)$, where c is an element of \mathcal{M}_a . Then there is a Skolem term $t(x)$ such that $\mathcal{N} \models c < t(a)$. Consider the following set:

$$Y = \{z \in N : (\mathcal{N}, X) \models \exists y < t(a) \forall x < z \phi(x, y) \leftrightarrow x \in X\}$$

This set contains \mathcal{M}_a . It must also be bounded, since, if it were not, then $Y = N$, and there would be some $b < t(a)$ such that $\phi(\cdot, b)$ defines X , but X is undefinable. Let b be the maximum of Y . Clearly b is a definable element in (\mathcal{N}, X, a) , and is above \mathcal{M}_a . \square

Starting with a model \mathcal{M} that is a conservative extension of all its elementary cuts, and \mathcal{N} a superminimal conservative extension of \mathcal{M} , we let $c_0 = 0$. If X is any undefinable class of \mathcal{N} , and $\mathcal{M}_a = \sup^N(\text{Scl}(a))$, applying this lemma we get $c_1 > \mathcal{M}_0$ definable in (\mathcal{N}, X) , and continuing we get $c_0 < c_1 < c_2 < \dots$, a sequence of elements, definable in (\mathcal{N}, X) , such that $c_{i+1} > \mathcal{M}_{c_i}$ for each $i \in \omega$. For limit stages, we need the following lemma.

Lemma 3. *Suppose $\mathcal{N} \models \text{PA}$, X is an undefinable class of \mathcal{N} , and $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$. If $(c_\beta : \beta < \lambda)$ is a λ -sequence of elements cofinal in \mathcal{M} , then there is $c_\lambda > \mathcal{M}$ such that $c_\lambda \in \text{dcl}^{(\mathcal{N}, X)}(\{c_\beta : \beta < \lambda\})$.*

Proof. Because \mathcal{N} is a conservative extension of \mathcal{M} , we have $X \cap M = \{x \in M : \mathcal{M} \models \phi(x, b)\}$ for some $b \in M$. Because the sequence $(c_\beta : \beta < \lambda)$ is cofinal in \mathcal{M} , there is some $\beta < \lambda$ such that $b < c_\beta$. Similar to the proof

above, we consider the set

$$\{z \in N : (\mathcal{N}, X) \models \exists y < c_\beta \forall x < z (\phi(x, y) \leftrightarrow x \in X)\}$$

This set is bounded and contains M , so let c_λ be the maximum of this set. □

Let \mathcal{N} and \mathcal{M} be as in the statement of the Proposition, and let $X \subseteq N$ be an undefinable class of \mathcal{N} . Let α be the order type of the set of elementary cuts of \mathcal{M} ordered by inclusion. Let $c_0 = 0$. For each $\beta < \alpha$, let $c_{\beta+1} \in N$ be an element definable in $(\mathcal{N}, X, c_\beta)$ such that $c_{\beta+1} > \sup^N(\text{Scl}(c_\beta))$ as per Lemma 2. Given $(c_\beta : \beta < \lambda)$ for some limit $\lambda < \alpha$, let c_λ be an element definable in $(\mathcal{N}, X, c_\beta)_{\beta < \lambda}$ such that $c_\lambda > \sup^N(\text{Scl}(c_\beta : \beta < \lambda))$.

We now have an increasing sequence $(c_\gamma : \gamma < \beta)$, for some $\beta \leq \alpha$ cofinal in \mathcal{M} . All these elements are definable in (\mathcal{N}, X) . If α is a successor ordinal, we apply Lemma 2 again, otherwise we apply Lemma 3. In both cases, we obtain $c > \mathcal{M}$ definable in (\mathcal{N}, X) . By superminimality, $\text{Scl}(c) = \mathcal{N}$, so \mathcal{N} is Enayat. □

Next we show that there are Enayat models whose substructure lattices contain an infinite descending chain.

Proposition 2. *Let T be a completion of PA. There is an Enayat model $\mathcal{M} \models T$ such that $\text{Lt}(\mathcal{M}) = 1 + \omega^*$.*

Proof. First, we need to state Mills's Theorem ([Mil79]). This result uses end-extensional types, which were pioneered by Gaifman in [Gai76].

Definition 3. Let T be a completion of PA and $p(x)$ an unbounded type over T . $p(x)$ is *end-extensional* if, for any formula $\phi(u, x)$, there are a Skolem term $t(u)$ and a formula $\sigma(u)$ such that the formula

$$\forall u[t(u) < x \rightarrow (\sigma(u) \leftrightarrow \phi(u, x))]$$

is in $p(x)$.

Mills's Theorem on distributive lattices states that if D is any \aleph_1 -algebraic distributive lattice and T is any completion of PA , there is an end-extensional type $p(x)$ producing the substructure lattice D . That is, if $\mathcal{M}_T \models T$ is a minimal model, and $\mathcal{M}_T(a)$ is an elementary extension of \mathcal{M}_T generated by an element a realizing $p(x)$, then $\text{Lt}(\mathcal{M}_T(a)) \cong D$. Letting D be the lattice $1 + \omega^*$, we can show that, for any completion T of PA , there is a model $\mathcal{M} \models T$ whose substructure lattice is isomorphic to $1 + \omega^*$.

Let \mathcal{N} be a superminimal conservative extension of such a model \mathcal{M} . $\text{Lt}(\mathcal{N}) = 1 + \omega^*$, since any superminimal extension of a model whose substructure lattice is $1 + \omega^*$ also has substructure lattice $1 + \omega^*$. Because the type is end extensional, if \mathcal{M}_T is the prime model of T , then $\mathcal{M}_T \prec_{\text{cons}} \mathcal{M}$. This is not hard to see: let $X = \{u \in M : \mathcal{M} \models \phi(u, a)\}$ where a is an element in \mathcal{M} realizing $p(x)$. Then, since $p(x)$ is end-extensional, there is a formula $\sigma(u)$ such that $X \cap M_T$ is defined by $\{u : \mathcal{M}_T \models \sigma(u)\}$.

We have that $\mathcal{M}_T \prec_{\text{cons}} \mathcal{M} \prec_{\text{cons}} \mathcal{N}$. Further, \mathcal{M}_T and \mathcal{M} are the only elementary cuts of \mathcal{N} . To see this, we appeal to the end-extensionality of $p(x)$. Gaifman [Gai76, Theorem 2.21] states that if $p(x)$ is end-extensional and $t(u, x)$ is some Skolem term, then there are Skolem terms $t_0(u)$, $t_1(u)$ and $t_2(y)$ such that the following formula is in $p(x)$:

$$\forall u [x > t_0(u) \rightarrow (t(u, x) = t_1(u) \vee t_2(t(u, x)) \geq x) \quad (2.1)$$

Let $a \in M$ realize $p(x)$ and suppose $b \in M \setminus M_T$. Then, there must be $m \in M_T$ such that $\mathcal{M} \models t(m, a) = b$. Letting t_0, t_1 , and t_2 be as in the statement (2.1), we know that $\mathcal{M} \models t_2(b) \geq a$ and therefore there is no proper elementary cut of \mathcal{M} containing b .

By Proposition 1, \mathcal{N} is Enayat. □

In some sense, this construction is not inherently different from other Enayat models that we have constructed. We start with some “base” and then build superminimal, conservative extensions on top and get an Enayat model. The question is if there are any other truly different constructions. To answer this, we look for a model whose elementary cuts form an infinite descending chain. In the language of ranked lattices, we look for a model \mathcal{M} such that $\text{Ltr}(\mathcal{M}) = (1 + \omega^*, \text{id})$; that is, we want the rank function of the substructure lattice of \mathcal{M} to be the identity.

First, we show that if a model \mathcal{M} is such that $\text{Lt}(\mathcal{M}) = 1 + \omega^*$, and for all $\mathcal{K} \in \text{Lt}(\mathcal{M})$, $\mathcal{K} \prec_{\text{cons}} \mathcal{M}$, then it must be Enayat. In particular, since

conservative extensions are always end extensions, this implies that all of the elementary substructures of \mathcal{M} are cuts.

Proposition 3. *Let $\mathcal{M} \models \text{PA}$ be such that $\text{Lt}(\mathcal{M}) = 1 + \omega^*$, and for all $\mathcal{K} \in \text{Lt}(\mathcal{M}), \mathcal{K} \prec_{\text{cons}} \mathcal{M}$. If X is an undefinable class of \mathcal{M} , then every element of \mathcal{M} is definable in (\mathcal{M}, X) .*

Proof. Let \mathcal{M}_0 be the minimal submodel of \mathcal{M} . Because $\text{Lt}(\mathcal{M}) = 1 + \omega^*$, all other submodels of \mathcal{M} are finitely generated. Let c_n , for $n \in \omega$, be such that $\text{Scl}(c_0) = \mathcal{M}$ and $\text{Scl}(c_{i+1}) \prec \text{Scl}(c_i)$. Let X be an undefinable class of \mathcal{M} . Because \mathcal{M} is conservative over all its submodels, we have formulas $\phi_i(x, c_i)$ that define $X \cap \text{Scl}(c_i)$ in $\text{Scl}(c_i)$ for each $i > 0$. We also have a formula $\phi(x)$ (without parameters) which defines $X \cap \mathcal{M}_0$ in \mathcal{M}_0 . Using these formulas, we get that c_n is definable from $(\mathcal{M}, X, c_{n+1})$ for every $n \in \omega$ (notice that c_n is not definable from c_{n+1} without X): let a be the least element such that $\neg(\phi_{n+1}(a, c_{n+1}) \iff a \in X)$ holds. Then $a > \text{Scl}(c_{n+1})$.

In fact, this shows that $\text{Scl}^{(\mathcal{M}, X)}(c_n) = \mathcal{M}$, as we can continue this process until we define c_0 , which generates all of \mathcal{M} .

Knowing this, we only need to show that there is some $n \in \omega$ such that c_n is definable from (\mathcal{M}, X) . Because X is not definable, there is $y \in \mathcal{M}$ such that $\mathcal{M} \models \neg(y \in X \iff \phi(y))$. Let a be the least such element. Then a must be above \mathcal{M}_0 , so it is in $\text{Scl}(c_n)$ for some n . In fact, $\text{Scl}(a) = \text{Scl}(c_n)$ for some n , since every elementary submodel of \mathcal{M} is generated by some c_n . Hence, since a is definable in (\mathcal{M}, X) , so must c_n , and we are done. \square

Showing that a model satisfying the assumptions of Proposition 3 exists requires more work.

Theorem 1. *Let T be a completion of PA, and let \mathcal{M}_T be a prime model of T . There is $\mathcal{M} \succ \mathcal{M}_T$ such that $\text{Lt}(\mathcal{M}) = 1 + \omega^*$ and for all $\mathcal{K} \in \text{Lt}(\mathcal{M})$, $\mathcal{K} \prec_{\text{cons}} \mathcal{M}$.*

Proof. The idea is to construct a sequence of representations of the n -element chains. $\alpha_1 : 2 \rightarrow \text{Eq}(M_T)$ is given by the definition of representation. If $\alpha : n \rightarrow \text{Eq}(A)$ is a representation, we say $\beta : n + 1 \rightarrow \text{Eq}(A)$ extends α if for all $0 < j < n$, $\alpha(j) = \beta(j + 1)$. We will use the following lemmas for our construction.

Lemma 4. *Let $\alpha : n \rightarrow \text{Eq}(A)$ be a representation such that $n > 1$ and $\alpha(1)$ has unboundedly many classes. Then there is a representation $\beta : n + 1 \rightarrow \text{Eq}(A)$ such that $\beta(1)$ has unboundedly many classes and β extends α .*

Proof. We say a sequence $(X_i : i \in M_T)$ of sets is a *definable enumeration* in \mathcal{M}_T if there is a formula $\phi(x, y)$ such that $X_i = \{x : \mathcal{M}_T \models \phi(x, i)\}$. Let X_0, X_1, \dots be a definable enumeration of the $\alpha(1)$ classes of A . Define $\beta : n + 1 \rightarrow \text{Eq}(A)$ as follows:

$$\beta(0) = A^2$$

$$\beta(j + 1) = \alpha(j), \text{ if } j > 0$$

$$\beta(1) = \{(x, y) : \exists n, m, i (x \in X_{\langle n, i \rangle} \text{ and } y \in X_{\langle m, i \rangle})\}$$

Then by definition β satisfies $\beta(j+1) = \alpha(j)$ for each $0 < j < n$. β is clearly a representation because if $(x, y) \in \beta(j)$ for some $j > 1$, then $(x, y) \in \alpha(j-1) \implies (x, y) \in \alpha(1) \implies (x, y) \in \beta(1)$.

We lastly need to check that $\beta(1)$ has unboundedly many classes, but this is clear from the definition, and that there are unboundedly many different X_i . \square

Since α_1 is already defined, Lemma 4 gives us representations $\alpha_n : n + 1 \rightarrow \text{Eq}(M_T)$ for each $1 \leq n \in \omega$. Here we note that for each n , if we let $\beta_n : n + 1 \rightarrow \text{Eq}(M_T^n)$ be defined by

$$\forall j \leq i (\langle x_1, \dots, x_n \rangle, \langle y_1, \dots, y_n \rangle) \in \beta_n(i) \iff x_j = y_j$$

then $\alpha_n \cong \beta_n$ (see Definition 2). We will show that $\alpha_2 \cong \beta_2$; the general argument is similar. We map $\langle x, y \rangle \in M_T^2$ to the y -th element of x -th $\alpha_2(1)$ class. Since every element of M_T is in some $\alpha_2(1)$ class, and each $\alpha_2(1)$ class has unboundedly many elements, this map is a bijection (and in fact is definable in \mathcal{M}_T). For $n > 2$, we note that each $\alpha_n(i)$ class contains unboundedly many $\alpha_n(i+1)$ classes, and so we can define an isomorphism in a similar fashion.

The representations α_n satisfy a canonicity property defined in the next lemma.

Lemma 5. *Let $\Theta \in \text{Eq}(M_T)$ be definable and let $n \geq 1$. Suppose $A \in \text{Def}(\mathcal{M}_T)$ is such that $\alpha_n|_A \cong \alpha_n$. Then there is a definable $B \subseteq A$ and $i \leq n$ such that $\Theta \cap B^2 = \alpha_n(i) \cap B^2$ and $\alpha_n|_B \cong \alpha_n$.*

Proof. Let n and Θ be given. This proof is by induction on n . If $n = 1$, then since A must be unbounded, we can find an unbounded subset B on which Θ is trivial or discrete.

Let $n > 1$ and assume that the lemma holds for α_{n-1} . Notice that if X is an $(\alpha_n|A)(1)$ class, then $\alpha_{n-1}|X \cong \alpha_{n-1}$. So for each such class X , by the inductive assumption, we know there is some $Y \subseteq X$ and $j < n$ such that $\alpha_{n-1}(j) \cap Y^2 = \Theta \cap Y^2$ and $\alpha_{n-1}|Y \cong \alpha_{n-1}$.

Let $(X_i : i \in M_T)$ be a definable enumeration of the $(\alpha_n|X)(1)$ classes. Then for each $i \in M_T$ there are $k_i < n$ and $Y_i \subseteq X_i$ such that $\alpha_{n-1}(k_i) \cap Y_i^2 = \Theta \cap Y_i^2$ and $\alpha_{n-1}|Y_i \cong \alpha_{n-1}$. Since, for each $i \in M_T, k_i < n$, there must be some $k < n$ such that there are unboundedly many i such that $\alpha_{n-1}(k) \cap Y_i^2 = \Theta \cap Y_i^2$. For the sake of simplicity, we will only consider those Y_i such that $\Theta \cap Y_i^2 = \alpha_{n-1}(k) \cap Y_i^2$ (so, in the enumeration below, we are not including any other Y_i 's).

If $k = 0$, then there is an unbounded $I \subseteq M_T$ such that

$$\forall i, j \in I, x \in Y_i, y \in Y_j, (x, y) \in \Theta \vee \forall i \neq j \in I, x \in Y_i, y \in Y_j, (x, y) \notin \Theta$$

Let $B = \bigcup_{i \in I} Y_i$. Either $\alpha_n(0) \cap B^2 = \Theta \cap B^2$ or $\alpha_n(1) \cap B^2 = \Theta \cap B^2$. In both cases, $\alpha_n|B \cong \alpha_n$.

If $k > 0$, we enumerate (in M_T) the Y_i as Z_0, Z_1, \dots so that each Y_i appears unboundedly often. For each class Z_i , there are $\alpha_{n-1}(1)$ classes Z_i^j (for each $j \in M_T$). Notice that since $k > 0$, for a fixed i , and for $j \neq j'$, if $x \in Z_i^j$

and $y \in Z_i^{j'}$, then $(x, y) \notin \Theta$. Hence for each fixed Z_i , there are unboundedly many Θ classes.

Let $i_0 = \langle 0, 0 \rangle$. Then given i_0, \dots, i_m , let $i_{m+1} = \langle m+1, j \rangle$, where j is the least such that for all $\langle i, j' \rangle \in \{i_0, \dots, i_m\}$, if $x \in Z_i^{j'}$ and $y \in Z_{m+1}^j$, then $(x, y) \notin \Theta$. Let $I = \{i_m : m \in M_T\}$, and $B = \bigcup_{\langle i, j \rangle \in I} Z_i^j$. Since B has unboundedly many $\alpha_n(1)$ classes, and for each such class X , we have that $\alpha_{n-1}|X \cong \alpha_{n-1}$, it follows that $\alpha_n|B \cong \alpha_n$. And since for all $x, y \in B$, we have

$$(x, y) \in \Theta \iff (x, y) \in \alpha_{n-1}(k) \iff (x, y) \in \alpha_n(k+1)$$

(since $k > 0$), we are done. \square

This next lemma will be used to ensure that we construct a definable type.

Lemma 6. *Let $\phi(u, v)$ be any formula. Suppose $A \in \text{Def}(\mathcal{M}_T)$ is such that, for some $n \geq 2$, $\alpha_n|A \cong \alpha_n$. Then, there is a definable $E \subseteq A$ such that $\alpha_n|E \cong \alpha_n$ and, letting $X = \{v \in E : \forall x < v (x \in E \rightarrow (x, v) \notin \alpha_n(1))\}$,*

$$\begin{aligned} \mathcal{M}_T \models & \forall u [\exists w \forall v \in X (v > w \wedge \forall x \in E (x, v) \in \alpha_n(1)) \rightarrow \phi(u, x)] \\ & \vee [\exists w \forall v \in X (v > w \wedge \forall x \in E (x, v) \in \alpha_n(1)) \rightarrow \neg \phi(u, x)] \end{aligned}$$

Proof. In other words, we are looking to find $E \subseteq A$ such that $\alpha_n|E \cong \alpha_n$ and, for each u , for all but finitely many $(\alpha_n|E)(1)$ classes, and for all x in those classes, $\phi(u, x)$ holds, or for all but finitely many $(\alpha_n|E)(1)$ classes and for all x in those classes, $\neg \phi(u, x)$ holds.

Let $n = 2$, and $\phi(u, v)$ a formula with two free variables. Let X_0, X_1, \dots be an enumeration of the (unboundedly many) $(\alpha_2|A)(1)$ classes. For $u \in M_T$, define $A_u = \{x \in A : \mathcal{M}_T \models \phi(u, x)\}$, $B_u = \{x \in A : \mathcal{M}_T \models \neg\phi(u, x)\}$.

We will define sets $I_0 \supseteq I_1 \supseteq \dots$ and $J_0 \supseteq J_1 \supseteq \dots$ as follows. Let

$$I_0^0 = \{i \in M_T : X_i \cap A_0 \text{ is unbounded} \},$$

$$I_0^1 = \{i \in M_T : X_i \cap B_0 \text{ is unbounded} \}$$

Then one of these must be unbounded, so define I_0 as below:

$$I_0 = \begin{cases} I_0^0 & \text{if } I_0^0 \text{ is unbounded} \\ I_0^1 & \text{otherwise} \end{cases}$$

Let $J_0 = A_0$ if $I_0 = I_0^0$, or B_0 otherwise.

Given I_k and J_k , define:

$$I_{k+1}^0 = \{i \in I_k : X_i \cap J_k \cap A_{k+1} \text{ is unbounded} \}$$

$$I_{k+1}^1 = \{i \in I_k : X_i \cap J_k \cap B_{k+1} \text{ is unbounded} \}$$

and define I_{k+1} to be I_{k+1}^0 if this set is unbounded, or I_{k+1}^1 otherwise. Let $J_{k+1} = J_k \cap A_{k+1}$ if $I_{k+1} = I_{k+1}^0$, or $J_k \cap B_{k+1}$ otherwise. Notice that for every k , all the I_k and J_k are unbounded, and if $i \in I_k$, then $J_k \cap X_i$ is unbounded.

Let $i_0 = \min(I_0)$, and if i_0, \dots, i_k are defined, let $i_{k+1} = \min(I_{k+1} \setminus$

$\{i_0, \dots, i_k\}$). Then let $I = \{i_k : k \in M_T\}$. We will use this set I to determine what the $\alpha_2(1)$ classes of E will be.

Let $Y_{i_k} = X_{i_k} \cap J_k$ for each $k \in M_T$, and let $E = \bigcup_{i \in I} Y_i$. Then there are unboundedly many $(\alpha_2|E)(1)$ classes, and each class is unbounded, so $\alpha_2|E \cong \alpha_2$. Further, given $u \in M_T$, for any $m \geq u$, it is clear that either $\mathcal{M}_T \models \forall x \in Y_{i_m} \phi(u, x)$ or $\mathcal{M}_T \models \forall x \in Y_{i_m} \neg \phi(u, x)$.

Inductively assume the lemma holds for n . Enumerate all $(\alpha_n|A)(1)$ classes X_0, X_1, \dots , and then notice that for each i , $\alpha_{n-1}|X_i \cong \alpha_{n-1}$. By the inductive hypothesis, there is $Y_i \subseteq X_i$ for each i such that $\alpha_{n-1}|Y_i \cong \alpha_{n-1}$. For a fixed i , let Y_i^0, Y_i^1, \dots be an enumeration of the $(\alpha_{n-1}|Y_i)(1)$ classes. By the inductive hypothesis, we know that for each $u \in M_T$, and for each i , there is w such that for all $m > w$, either $\mathcal{M}_T \models \forall x \in Y_i^m \phi(u, x)$ or $\mathcal{M}_T \models \forall x \in Y_i^m \neg \phi(u, x)$.

Let

$$I_0^0 = \{i \in M_T : \mathcal{M}_T \models \exists w \forall m > w \forall x \in Y_i^m \phi(0, x)\},$$

$$I_0^1 = \{i \in M_T : \mathcal{M}_T \models \exists w \forall m > w \forall x \in Y_i^m \neg \phi(0, x)\}$$

As above, let $I_0 = I_0^0$ if that is unbounded, or I_0^1 otherwise (in either case, I_0 is unbounded).

If I_k is defined, let

$$I_{k+1}^0 = \{i \in I_k : \mathcal{M}_T \models \exists w \forall m > w \forall x \in Y_i^m \phi(k, x)\}$$

$$I_{k+1}^1 = \{i \in I_k : \mathcal{M}_T \models \exists w \forall m > w \forall x \in Y_i^m \neg \phi(k, x)\}$$

We define $I_{k+1} = I_{k+1}^0$ if this set is unbounded, or $I_{k+1} = I_{k+1}^1$ otherwise, and either way I_{k+1} will be unbounded.

Let $i_0 = \min(I_0)$, and if i_0, \dots, i_k are defined, let $i_{k+1} = \min(I_{k+1} \setminus \{i_0, \dots, i_k\})$. Then let $I = \{i_k : k \in M_T\}$. Again, this will determine our $\alpha_n(1)$ classes (that is, which Y_i we will end up using). For a fixed u and for $i, j \in I$ large enough, Y_i and Y_j will “agree” on the truth of $\phi(u, x)$ in the sense that if m_1, m_2 are large enough, and $x \in Y_i^{m_1}, y \in Y_j^{m_2}$ then $\phi(u, x) \leftrightarrow \phi(u, y)$. The goal is to collect these Y_i^j that agree in this sense.

If $I_0 = I_0^0$, then given $i \in I$, there is w such that for all $m > w$ and all $x \in Y_i^m$, $\phi(0, x)$. If $I_0 = I_0^1$, and $i \in I$, there is w such that for all $m > w$ and all $x \in Y_i^m$, $\neg\phi(0, x)$. In either case, let w_0 be the witness to either of the above statements for $i = i_0$, and let $Z_{i_0} = \bigcup_{m > w_0} Y_{i_0}^m$. Similarly, if $I_k = I_k^0$ then for each $i \geq i_k \in I$, there is w such that for all $m > w$ and $x \in Y_i^m$, $\phi(k, x)$ holds, and if $I_k = I_k^1$, for $i \geq i_k \in I$, there is w such that for all $m > w, x \in Y_i^m$, $\neg\phi(k, x)$ holds. There is also w' such that for all $m > w', x \in Y_i^m, j < k, u < k, y \in Z_{i_j}, \phi(u, x) \leftrightarrow \phi(u, y)$. Let w_k be the maximum of these w (for whichever statement holds) and w' , and then let $Z_{i_k} = \bigcup_{m > w_k} Y_{i_k}^m$.

Let $E = \bigcup_{i \in I} Z_i$. First, $\alpha_n|E \cong \alpha_n$. We see this because there are unboundedly many $(\alpha_n|E)(1)$ classes, and each $(\alpha_n|E)(1)$ class is Z_i for some $i \in I$, and $\alpha_{n-1}|Z_i \cong \alpha_{n-1}|Y_i$, since Z_i contains cofinitely many Y_i^j .

Let $u \in M_T$. If $k > u$, then either $\mathcal{M}_T \models \forall x \in Z_{i_k} \phi(u, x)$ or $\mathcal{M}_T \models \forall x \in Z_{i_k} \neg\phi(u, x)$, so the theorem holds. \square

Notice in the above proofs, for $n > 1$, given a set $B \subseteq M_T$, we conclude that if there are unboundedly many $(\alpha_n|B)(1)$ classes X such that $\alpha_{n-1}|X \cong \alpha_{n-1}$, then $\alpha_n|B \cong \alpha_n$. We can generalize this: given a set B , if X is a set of representatives of $(\alpha_n|B)(r)$ classes, and $\alpha_n|X$ “looks like” α_r , and each $(\alpha_n|B)(r)$ class “looks like” α_{n-r} , then $\alpha_n|B \cong \alpha_n$.

Lemma 7. *Let $n > 1$ and $0 < r < n$. Let $B \in \text{Def}(\mathcal{M}_T)$ and $X = \{x \in B : \mathcal{M}_T \models \forall y < x (x, y) \notin \alpha_n(r)\}$. If $(\alpha_n|X) \upharpoonright r + 1 \cong \alpha_r$ and for each $(\alpha_n|B)(r)$ class Y , $\alpha_{n-r}|Y \cong \alpha_{n-r}$, then $\alpha_n|B \cong \alpha_n$.*

Proof. We will prove this lemma by making use of the representations $\beta_n : n + 1 \rightarrow \text{Eq}(M_T^n)$ defined above. We will define a bijection $h : B \rightarrow M_T^n$ confirming an isomorphism $\alpha_n|B \cong \beta_n$.

Fix a bijection $f : X \rightarrow M_T^r$ that confirms the isomorphism $(\alpha_n|X) \upharpoonright r + 1 \cong \beta_r$. For a given $x \in B$, let

$$Y = \{y \in B : (x, y) \in \alpha_n(r)\}$$

Fix a bijection $g : Y \rightarrow M_T^{n-r}$ confirming the isomorphism $\alpha_{n-r}|Y \cong \beta_{n-r}$. Let x' be the least element of Y (so $x' \in X$). If $f(x') = (x_1, \dots, x_r)$, and $g(x) = (x_{r+1}, \dots, x_n)$, then let $h(x) = (x_1, \dots, x_r, x_{r+1}, \dots, x_n)$. This is a bijection, and we will show that for all $x, y \in B$ and $k \leq n$

$$(x, y) \in \alpha_n(k) \iff (h(x), h(y)) \in \beta_n(k)$$

Let $x, y \in B$. Let x' and y' be the least elements of the $\alpha_n(r)$ classes of x and y , respectively. Let $h(x) = (x_1, \dots, x_n)$ and $h(y) = (y_1, \dots, y_n)$, so $f(x') = (x_1, \dots, x_r)$ and $f(y') = (y_1, \dots, y_r)$. Let Y_x and Y_y be the $\alpha_n(r)$ classes of x and y respectively, and let g_x and g_y be the bijections confirming the isomorphisms $\alpha_{n-r}|Y_x \cong \beta_{n-r}$ and $\alpha_{n-r}|Y_y \cong \beta_{n-r}$ respectively. In the cases where x and y are in the same α_{n-r} class, we will simply call this function g .

Suppose $(x, y) \in \alpha_n(k)$, and $k \leq r$. Then $(x, x') \in \alpha_n(k)$ and $(y, y') \in \alpha_n(k)$, so $(x', y') \in \alpha_n(k)$, and therefore $(f(x'), f(y')) \in \beta_r(k)$. Therefore $x_j = y_j$ for all $j \leq k$, and then $(h(x), h(y)) \in \beta_n(k)$.

Now suppose $k > r$. Then $(x, y) \in \alpha_n(r)$ so $x' = y'$ and therefore $x_j = y_j$ for all $j \leq r$. Also,

$$\begin{aligned} \forall r < j \leq k [(x, y) \in \alpha_n(k) \implies (x, y) \in \alpha_{n-r}(k-r) \implies \\ (g(x), g(y)) \in \beta_{n-r}(k-r) \implies x_j = y_j] \end{aligned}$$

Therefore $(h(x), h(y)) \in \beta_n(k)$.

Conversely, suppose $(x, y) \notin \alpha_n(k)$ and again first suppose $k \leq r$. Then,

$$(x', y') \notin \alpha_n(k) \implies (f(x'), f(y')) \notin \beta_r(k)$$

Therefore there is some $j \leq k$ such that $x_j \neq y_j$, and so $(h(x), h(y)) \notin \beta_n(k)$.

Now if $k > r$, then

$$(x, y) \notin \alpha_n(k) \implies (x, y) \notin \alpha_{n-r}(k-r)$$

There are two possibilities: either x and y are in the same $\alpha_n(r)$ class Y , or they are not. If they are not, then $(x', y') \notin \alpha_n(r)$, and therefore $(f(x'), f(y')) \notin \beta_n(r)$, so there is some $j \leq r$ such that $x_j \neq y_j$. Therefore $(h(x), h(y)) \notin \alpha_n(r)$, and since $r < k$, $(h(x), h(y)) \notin \alpha_n(k)$.

Finally, if $(x, y) \in \alpha_n(r)$ but $(x, y) \notin \alpha_n(k)$, then

$$(x, y) \notin \alpha_{n-r}(k-r) \implies (g(x), g(y)) \notin \beta_{n-r}(k-r)$$

Therefore there is some j such that $r < j \leq k$ such that $x_j \neq y_j$, so $(h(x), h(y)) \notin \alpha_n(k)$. \square

We will construct a sequence of representations isomorphic to the α_n defined above, which we will again call $\alpha_n : n + 1 \rightarrow \text{Eq}(A_n)$. For a given $i \in \omega$, if α_{i+1} and A_{i+1} are defined, then we will let t_i be the term defined on A_{i+1} mapping an $x \in A_{i+1}$ to the least $y \in A_{i+1}$ such that $(x, y) \in \alpha_{i+1}(1)$. We will define these A_n later. Given this definition, we can state the next lemma, which will be used to ensure that we end up with a model that is conservative over all its submodels.

Lemma 8. *Let $n \geq i + 2$, and $\phi(u, x)$ a formula. Assume t_{i-1} and t_i are defined and $A \in \text{Def}(\mathcal{M}_T)$ is such that $\alpha_n|A \cong \alpha_n$. Then there is a definable*

$B \subseteq A$ such that $\alpha_n|B \cong \alpha_n$ and the following holds in \mathcal{M}_T :

$$\begin{aligned} \forall v \in B \forall u [\exists w \forall x \in B (t_i(x) = t_i(v) \wedge t_{i-1}(x) > w) \rightarrow \phi(u, t_{i-1}(x))] \\ \forall [\exists w \forall x \in B (t_i(x) = t_i(v) \wedge t_{i-1}(x) > w) \rightarrow \neg \phi(u, t_{i-1}(x))] \end{aligned}$$

Proof. Similar to Lemma 6, we are looking to find a definable set $B \subseteq A$ such that $\alpha_n|B \cong \alpha_n$ and, for each $(\alpha_n|B)(n-i)$ class X , for all u , either for all but finitely many $(\alpha_n|X)(n-i+1)$ classes and all x in those classes, $\phi(u, t_{i-1}(x))$ holds, or for all but finitely many $(\alpha_n|X)(n-i+1)$ classes and all x in those classes, $\neg \phi(u, t_{i-1}(x))$ holds.

The rough idea for this is that for each $\alpha_n(n-i)$ class C , we will let $D = \{t_{i-1}(x) : x \in C\}$ and apply Lemma 2.2.4 in [KS06], which shows that there is a definable, unbounded $E \subseteq D$ such that the following holds in \mathcal{M}_T :

$$\begin{aligned} \forall u [\exists w \forall x > w (x \in E \rightarrow \phi(u, x))] \\ \forall [\exists w \forall x > w (x \in E \rightarrow \neg \phi(u, x))] \end{aligned}$$

Then we will let $C' = \{x \in C : t_{i-1}(x) \in E\}$. Collect all such C' into a set B , and then B should have the required properties, but we must be careful to ensure we end up with a definable B .

Let $X = \{x \in A : \forall v < x (x, v) \notin \alpha_n(n-i)\}$. This is an unbounded definable set, so let x_k be the k^{th} element of X , so that $(x_k : k \in M_T)$ enumerates X . We will define sets $I_{\langle k, m \rangle}$ and elements $e_{\langle k, m \rangle}$ so that $I_{\langle k, m \rangle} \subseteq$

$I_{\langle k, m+1 \rangle}$ for each $k, m \in M_T$.

Let $C_k = \{y \in A : (x_k, y) \in \alpha_n(n-i)\}$, and $D_k = \{t_{i-1}(x) : x \in C_k\}$. Each D_k is unbounded. Let $X_k^u = \{x \in D_k : \mathcal{M}_T \models \phi(u, x)\}$ and let $Y_k^u = \{x \in D_k : \mathcal{M}_T \models \neg\phi(u, x)\}$.

We define $I_{\langle k, 0 \rangle}, e_{\langle k, 0 \rangle}$ similarly to how I_0 and e_0 are defined in Lemma 2.2.4 in [KS06]:

$$I_{\langle k, 0 \rangle} = \begin{cases} \{0\} & \text{if } X_k^0 \text{ is unbounded} \\ \emptyset & \text{otherwise} \end{cases}$$

Let $e_{\langle k, 0 \rangle}$ be the least element of X_k^0 if $0 \in I_{\langle k, 0 \rangle}$, and the least element of Y_k^0 otherwise.

If $I_{\langle k, m-1 \rangle}$ and $e_{\langle k, 0 \rangle}, \dots, e_{\langle k, m-1 \rangle}$ are defined, we define $I_{\langle k, m \rangle}$ and $e_{\langle k, m \rangle}$ (again, similarly to how I_m and e_m are defined in Lemma 2.2.4 in [KS06]):

$$I_{\langle k, m \rangle} = \begin{cases} I_{\langle k, m-1 \rangle} \cup \{m\} & \text{if } \bigcap (X_k^j : j \in I_{\langle k, m-1 \rangle}) \cap X_k^m \\ & \text{is unbounded} \\ I_{\langle k, m-1 \rangle} & \text{otherwise} \end{cases}$$

And let $e_{\langle k, m \rangle}$ be the least element of $\bigcap (X_k^j : j \in I_{\langle k, m \rangle}) \setminus \{e_{\langle k, 0 \rangle}, \dots, e_{\langle k, m-1 \rangle}\}$ if $m \in I_{\langle k, m \rangle}$, and the least element of $\bigcap (X_k^j : j \in I_{\langle k, m-1 \rangle}) \cap Y_k^m \setminus \{e_{\langle k, 0 \rangle}, \dots, e_{\langle k, m-1 \rangle}\}$ otherwise.

Let $E = \{e_m : m \in M_T\}$ and $B = \{x \in A : t_{i-1}(x) \in E\}$. First we will show, using Lemma 7, that $\alpha_n|B \cong \alpha_n$.

Let $X' = \{x \in B : \forall v \in B (v < x \rightarrow (x, v) \notin \alpha_n(n-i))\}$. If $x \in X'$, then $t_{i-1}(x) = e_{\langle k, m \rangle}$ for some k, m , and therefore $(x, x_k) \in \alpha_n(n-i)$. In addition, for any other $x_j \in X$, there is $v \in B$ such that $(x_j, v) \in \alpha_n(n-i)$, since we can let v be such that $t_{i-1}(v) = e_{\langle j, 0 \rangle}$. Let x be the least such v , so $x \in X'$ and $(x, x_j) \in \alpha_n(n-i)$. This shows that $(\alpha_n \upharpoonright n-i+1)|X' \cong (\alpha_n \upharpoonright n-i+1)|X$ and therefore $(\alpha_n \upharpoonright n-i+1)|X' \cong \alpha_{n-i}$.

Let C be an $(\alpha_n|B)(n-i)$ class. Then there is some k such that $C \subseteq C_k$. We wish to show that $\alpha_i|C \cong \alpha_i|C_k$. Let $f : A \rightarrow M_T^n$ be the bijection confirming the isomorphism $\alpha_n|A \cong \beta_n$. Let $x, y \in C_k$, and suppose $f(x) = \langle x_1, \dots, x_n \rangle$ and $f(y) = \langle y_1, \dots, y_n \rangle$. Then $x_j = y_j$ for all $j \leq n-i$, so the image under f of all $x \in C_k$ will have the same first $n-i$ components, which we will call x_1, \dots, x_{n-i} . Notice that if $x \in C$ and $y \in C_k$ are such that $(x, y) \in \alpha_n(n-i+1)$, then $t_{i-1}(x) = t_{i-1}(y)$ so $y \in C$ as well. Because f is a bijection, we have

$$\mathcal{M}_T \models \forall x_{n-i+1} \dots \forall x_n \exists! x \in C f(x) = \langle x_1, \dots, x_{n-i}, x_{n-i+1}, \dots, x_n \rangle$$

We will define $g : C \rightarrow C_k$ as follows. Let $Y \subseteq M_T$ be the set of all those j such that

$$\mathcal{M}_T \models \exists y_1 \dots \exists y_{i-1} \exists x \in C f(x) = \langle x_1, \dots, x_{n-i}, j, y_1, \dots, y_{i-1} \rangle$$

This set must be unbounded, since the function t_{i-1} has unbounded range on C (there are unboundedly many $\alpha_n(n-i+1)$ classes in each $\alpha_n(n-i)$ class). Enumerate the elements of Y as $j_0 < j_1 < \dots$. Let $y \in C$ and suppose $f(y) = \langle x_1, \dots, x_{n-i}, j_m, y_1, \dots, y_{i-1} \rangle$, and suppose $z \in C_k$ is such that $f(z) = \langle x_1, \dots, x_{n-i}, m, y_1, \dots, y_{i-1} \rangle$. Then let $g(y) = z$, and $g : C \rightarrow C_k$ confirms the isomorphism $\alpha_i|C \cong \alpha_i|C_k$: $(x, y) \in \alpha_i(r)$ iff $(x, y) \in \alpha_n(n-i+r)$ iff $(f(x), f(y)) \in \alpha_n(n-i+r)$ iff $(g(x), g(y)) \in \alpha_n(n-i+r)$ iff $(g(x), g(y)) \in \alpha_i(r)$. Since we can apply this procedure to find a g for each $(\alpha_n|B)(n-i)$ class C , by Lemma 7, $\alpha_n|B \cong \alpha_n$.

To finish the proof, let $v \in B$. There is some k such that $(x_k, v) \in \alpha_n(n-i)$. For each u , if $u \in I_{\langle k, u \rangle}$, then for all $m > u$, $\mathcal{M}_T \models \phi(u, e_{\langle k, m \rangle})$. Let $x \in B$ be such that $(x, v) \in \alpha_n(n-i)$. Then $t_{i-1}(x) \in D_k$ so it must be such an $e_{\langle k, m \rangle}$, so there is some w such that

$$\mathcal{M}_T \models \forall x \in B((x, v) \in \alpha_n(n-i) \wedge t_{i-1}(x) > w) \rightarrow \phi(u, t_{i-1}(x))$$

If $u \notin I_{\langle k, u \rangle}$, then there are only boundedly many m such that $\mathcal{M}_T \models \phi(u, e_{\langle k, m \rangle})$, and hence there is some w such that

$$\mathcal{M}_T \models \forall x \in B((x, v) \in \alpha_n(n-i) \wedge t_{i-1}(x) > w) \rightarrow \neg \phi(u, t_{i-1}(x))$$

□

Let $\theta_1, \theta_2, \dots$ be a recursive enumeration of formulas that define equivalence

relations on M_T such that each definable equivalence relation Θ on M_T is defined by infinitely many θ_i . Let $\phi_1(u, x), \phi_2(u, x), \dots$ be an enumeration of all formulas in \mathcal{L}_{PA} with two free variables. Enumerate all pairs of the form $\langle k, i \rangle$, with $k, i \in \omega, k \geq 1$ as $\langle k_1, i_1 \rangle, \langle k_2, i_2 \rangle, \dots$, so that for each $n \geq 1$, $i_n \leq n-1$ (note: the usual Cantor pairing function gives such an enumeration). Using Lemmas 4, 5, 6 and 8, we will construct a definable type $p(x)$.

At stage $n = 0$, let $A_0 = M_T$ and we have $\alpha_1 : 2 \rightarrow \text{Eq}(A_0)$.

At stage $s = 3n - 2$, for $n > 0$, we have $\alpha_n : n + 1 \rightarrow \text{Eq}(A_{n-1})$. Apply Lemma 5 to α_n and θ_n and we get a set B .

At stage $s = 3n - 1$, if $n = 1$, let $A_1 = A$ and continue. Otherwise, apply Lemma 6 on the formula $\phi_{n-1}(u, x)$ to get $A' \subseteq A$, and then apply Lemma 8 to $\phi_{k_{n-1}}$, with $i = i_{n-1}$, on $\alpha_n : n + 1 \rightarrow \text{Eq}(A')$ to get a set $B \subseteq A'$. Now let $A_n = B$.

At stage $s = 3n$, given $\alpha_n : n + 1 \rightarrow \text{Eq}(A_n)$, use Lemma 4 to construct $\alpha_{n+1} : n + 2 \rightarrow \text{Eq}(A_n)$.

This process results in a sequence of unbounded definable sets $M_T = A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$. Given n , let

$$X_n = \{v \in A_n : \forall x < v (x \in A_n \rightarrow (x, v) \notin \alpha_n(1))\}$$

Let $p(x)$ be the type

$$\{\phi_n(u, x) : \exists w \forall v \in X_n (v > w \rightarrow \forall y \in A_n ((y, v) \in \alpha_n(1) \rightarrow \phi(u, y)))\}$$

This type is complete by Lemma 6.

Let c realize $p(x)$ and let $\mathcal{M} = \mathcal{M}_T(c)$. Recall the definitions of the terms t_n given before: given $n \in \omega, x \in A_{n+1}, t_n(x)$ is the least element of x 's $\alpha_{n+1}(1)$ equivalence class. Let $c_n = t_n(c)$. We will show the following, which will prove the proposition:

1. $p(x)$ is a definable type.
2. $\mathcal{M}_T(c_j) \preceq \mathcal{M}_T(c_i) \iff i \leq j$ for every $i, j \in \omega$.
3. If $\mathcal{K} = \mathcal{M}_T(b)$ for some $b \in \mathcal{M}$, then there is some $n \in \omega$ such that

$$\mathcal{K} = \mathcal{M}_T(c_n)$$
4. $\mathcal{M}_T(c_{i+1}) \prec_{\text{end}} \mathcal{M}_T(c_i)$ for every $i \in \omega$
5. $\mathcal{M}_T(c_{i+1}) \prec_{\text{cons}} \mathcal{M}_T(c_i)$ for every $i \in \omega$

First, we show that $p(x)$ is a definable type. Given $\phi_n(u, x)$, let

$$\sigma_n(u) = \exists w \forall v \in X_n (v > w \rightarrow \forall y \in A_n (y, v) \in \alpha_n(1) \rightarrow \phi(u, y))$$

If $u \in M_T$ and $\mathcal{M} \models \phi_n(u, c)$, then $\phi_n(u, x) \in p(x)$, so $\mathcal{M}_T \models \sigma_n(u)$. Conversely, if $\mathcal{M}_T \models \sigma_n(u)$, then again by the definition of the type, $\phi_n(u, x) \in p(x)$ so $\mathcal{M} \models \phi_n(u, c)$.

$i \leq j \implies \mathcal{M}_T(c_j) \preceq \mathcal{M}_T(c_i)$: Notice that by construction, if $i < n$, we have $t_i(x) = t_i(y)$ if and only if $(x, y) \in \alpha_n(n - i + 1)$, for all $x, y \in A_n$. Therefore, if $i \leq j < n$, $t_i(x) = t_i(y) \implies t_j(x) = t_j(y)$ for all $x, y \in A_n$, since

$\alpha_n(n - i + 1) \subseteq \alpha_n(n - j + 1)$. Since $t_i(x) = t_i(t_i(x))$ for every $x \in A_n$, we have $t_j(x) = t_j(t_i(x))$ for all $x \in A_n$, which therefore implies that $t_j(c_i) = c_j$ whenever $i \leq j$. Hence $\mathcal{M}_T(c_j) \preceq \mathcal{M}_T(c_i)$.

$\mathcal{M}_T(c_j) \preceq \mathcal{M}_T(c_i) \implies i \leq j$: Let $t(x)$ be a Skolem term such that $\mathcal{M}_T(c_i) \models t(c_i) = c_j$. Therefore $t(t_i(x)) = t_j(x) \in p(x)$. Let θ be the equivalence relation given by $(x, y) \in \theta \iff t(x) = t(y)$. Let n be such that θ is canonical on A_n , and since $t(t_i(x)) = t_j(x) \in p(x)$, we can assume that for all $x \in A_n$, $\mathcal{M}_T \models t(t_i(x)) = t_j(x)$. For $x, y \in A_n$, if $t_i(x) = t_i(y)$, then $t_j(x) = t(t_i(x)) = t(t_i(y)) = t_j(y)$. Therefore, $\alpha_n(n - j) \supseteq \alpha_n(n - i)$; that is, $\alpha_n(n - i)$ refines $\alpha_n(n - j)$, so $n - i \geq n - j$ or $i \leq j$.

Suppose $b \in M$ and $\mathcal{K} = \mathcal{M}_T(b)$. We show that there is some n such that $\mathcal{K} = \mathcal{M}_T(c_n)$. Let $t(x)$ be a Skolem term such that $\mathcal{M} \models t(c) = b$. Let θ_n be the equivalence relation given by $x \sim y \iff t(x) = t(y)$ and assume it is canonical on A_n . Therefore there is $i < n$ such that $t(x) = t(y) \iff t_i(x) = t_i(y)$ for all $x, y \in A_n$. There must then be Skolem terms f and g such that $\forall x \in A_n f(t_i(x)) = t(x) \wedge g(t(x)) = t_i(x)$. We define these as follows: let $f(y) = t(x)$ where x is least such that $x \in A_n$ and $t_i(x) = t_i(y)$, and let $g(y) = t_i(x)$ where x is least such that $x \in A_n$ and $t(x) = y$ (or 0 if no such x exists). Therefore, since $\mathcal{M} \models f(c_i) = b \wedge g(b) = c_i$, we have that $\mathcal{M}_T(b) = \mathcal{M}_T(c_i)$.

Now since, for each i , $\mathcal{M}_T(c_{i+1}) \prec \mathcal{M}_T(c_i)$, and for each $b \in M$, there is some n such that $\mathcal{M}_T(b) = \mathcal{M}_T(c_n)$, $\text{Lt}(\mathcal{M}) \cong 1 + \omega^*$.

Now, we show that $\mathcal{M}_T(c_{i+1}) \prec_{\text{end}} \mathcal{M}_T(c_i)$ for $i \in \omega$. Let $f(x)$ and $g(x)$

be Skolem terms such that $f(t_i(c)) < g(t_{i+1}(c))$. We wish to show that there is a term $t(x)$ and an n such that for all $x \in A_n$ $t(t_{i+1}(x)) = f(t_i(x))$. We will in fact show that $f(t_i(x)) = f(t_{i+1}(x))$ for every $x \in A_n$.

If $i = 0$, this means that if $f(x) < g(t_1(x))$ then we must find a Skolem term $t(y)$ such that $t(t_1(x)) = f(x)$. We can further assume (by Lemma 5) that for $x, y \in A_n$, there is $j \leq n$ such that $f(x) = f(y) \iff t_j(x) = t_j(y)$. Notice that if $j \geq 1$, $t_j(x) = t_j(t_1(x)) \implies f(x) = f(t_1(x))$, so if $j \neq 0$ we are done. Assume $j = 0$ and fix $x \in A_n$. Let $m = g(t_1(x))$. There are unboundedly many $y \in A_n$ such that $t_1(x) = t_1(y)$, so we would have unboundedly many y such that $f(x) \neq f(y)$ but $f(y) < m$, which is impossible.

Let f, g , and n be such that for all $x \in A_n$, $f(t_i(x)) < g(t_{i+1}(x))$. We can again assume that for $x, y \in A_n$, there is $j \leq n$ such that $f(x) = f(y)$ if and only if $t_j(x) = t_j(y)$. If $j \geq i + 1$, then $t_j(x) = t_j(t_{i+1}(x))$ implies $f(x) = f(t_{i+1}(x))$. Further $t_j(x) = t_j(t_i(x))$ implies $f(x) = f(t_i(x)) = f(t_{i+1}(x))$, so again we are done in this case. If $j \leq i$, then $t_i(x) \neq t_i(y)$ implies $t_i(t_i(x)) \neq t_i(t_i(y))$. Therefore, $t_j(t_i(x)) \neq t_j(t_i(y))$, and so $f(t_i(x)) \neq f(t_i(y))$. Again we can find unboundedly many y such that $t_i(x) \neq t_i(y)$ but $t_{i+1}(x) = t_{i+1}(y)$, so if we let $m = g(t_{i+1}(x))$, we have unboundedly many values for $f(t_i(x))$ in A_n , while these values are all bounded by m .

Lastly, we show that $\mathcal{M}_T(c_{i+1}) \prec_{\text{cons}} \mathcal{M}_T(c_i)$. Let $X = \{u \in M_T(c_{i+1}) : \mathcal{M}_T(c_i) \models \phi(u, c_i)\}$. We claim that there is some n so that

$$X = \{u : \mathcal{M}_T(c_{i+1}) \models \exists w \forall x \in A_n (t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \implies \phi(u, t_i(x))\}$$

Suppose $u \in X$. Then $\mathcal{M} \models \phi(u, c_i)$ and

$$\begin{aligned} \mathcal{M} &\models \exists w \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \phi(u, t_i(x)) \\ &\forall \exists w \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \neg \phi(u, t_i(x)) \end{aligned}$$

By elementarity, this is also true in $\mathcal{M}_T(c_{i+1})$. Suppose

$$\mathcal{M}_T(c_{i+1}) \not\models \exists w \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \phi(u, t_i(x))$$

Then there is $w \in \mathcal{M}_T(c_{i+1})$ such that

$$\mathcal{M} \models \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \neg \phi(u, t_i(x))$$

Notice that $c \in A_n, t_{i+1}(c) = c_{i+1}$, and $c_i = t_i(c) > w$ since $w \in \mathcal{M}_T(c_{i+1})$.

Therefore $\mathcal{M} \models \neg \phi(u, c_i)$, which is a contradiction.

Now suppose $u \in \mathcal{M}_T(c_{i+1})$ is such that

$$\mathcal{M}_T(c_{i+1}) \models \exists w \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \phi(u, t_i(x))$$

Let $w \in \mathcal{M}_T(c_{i+1})$ be the witness in that statement, and therefore

$$\mathcal{M} \models \forall x \in A_n(t_{i+1}(x) = c_{i+1} \wedge t_i(x) > w) \rightarrow \phi(u, t_i(x))$$

Letting $x = c$, we get $\mathcal{M} \models \phi(u, c_i)$. □

2.4 Dolich Sets

Returning to the original question, we know now that in general, we cannot use arithmetic forcing to find inductive subsets X of a model \mathcal{M} such that the definable elements of (\mathcal{M}, X) coincide with the definable elements of \mathcal{M} . If \mathcal{M} is recursively saturated, this result follows from [Ena88, Theorems B and C], which Enayat proved for ZFC. The argument for PA is similar: use arithmetic forcing to obtain a generic set X_0 for $\text{Scl}(0)$, and then extend it to a generic set X so that $(\text{Scl}(0), X_0) \prec (\mathcal{M}, X)$.

Alfred Dolich posed a stronger question: given a countable, recursively saturated model $\mathcal{M} \models \text{PA}$, is there an undefinable, inductive set X such that the definable closure relation is unchanged? That is, is there an undefinable $X \subseteq M$ such that $(\mathcal{M}, X) \models \text{PA}^*$ and, for each $a, b \in M$, $a \in \text{Scl}^{(\mathcal{M}, X)}(b)$ if and only if $a \in \text{Scl}^{\mathcal{M}}(b)$? James Schmerl (via personal communication) answers this question in the negative:

Proposition 4. *If $\mathcal{M} \models \text{PA}$ is recursively saturated, there is no inductive, undefinable $X \subseteq M$ such that the dcl relation in (\mathcal{M}, X) coincides with that of \mathcal{M} .*

Proof. This proof is due to James Schmerl. If \mathcal{M} is recursively saturated and X is an inductive, undefinable subset of M such that the dcl relation in (\mathcal{M}, X) coincides with the dcl relation in \mathcal{M} , then, for each $\mathcal{K} \prec \mathcal{M}$, we have $(\mathcal{K}, K \cap X) \prec (\mathcal{M}, X)$. Let $\mathcal{M}_0 = \text{Scl}(0)$. We can find \mathcal{M}_1 such that $\mathcal{M}_0 \prec_{\text{cons}} \mathcal{M}_1 \prec \mathcal{M}$. This is possible because $\text{Th}(\mathcal{M}) \in \text{SSy}(\mathcal{M})$ and $\text{SSy}(\mathcal{M})$

is a Scott set. In general, if \mathfrak{X} is a Scott set, and $T \in \mathfrak{X}$ is a completion of PA , then there is a definable type $p(x)$ of T coded in \mathfrak{X} . This can be shown by defining a minimal type recursively in T . Let $a \in \mathcal{M}$ be a realization of $p(x)$ and let $\mathcal{M}_1 = \text{Scl}(a)$. Since $p(x)$ is definable, $\mathcal{M}_0 \prec_{\text{cons}} \mathcal{M}_1$. As a result, $M_0 \cap X$ is definable in \mathcal{M}_0 , so X is definable in \mathcal{M} by elementarity which is a contradiction. \square

We can weaken this property by ignoring the condition that X is inductive. In this case, we can find such sets: ω will be one, for example.

Proposition 5. *Let $\mathcal{M} \models \text{PA}$ be recursively saturated, and suppose $X \subseteq M$ is automorphism invariant. Then the dcl relation in (\mathcal{M}, X) coincides with the dcl relation in \mathcal{M} .*

Proof. Let $a, b \in M$ be such that $a \notin \text{dcl}(b)$. Then, since a is an undefinable element in (\mathcal{M}, b) , there is an automorphism f of \mathcal{M} fixing b such that $f(a) \neq a$. If $a \in \text{dcl}^{(\mathcal{M}, X)}(b)$, that means $(\mathcal{M}, X) \models \phi(a, b)$ for some ϕ in the expanded language. Also,

$$(\mathcal{M}, X) \models \forall x(\phi(x, b) \implies x = a)$$

Since $f \in \text{Aut}(\mathcal{M}, X)$, we have $(\mathcal{M}, X) \models \phi(f(a), b)$ and hence $f(a) = a$, which is a contradiction. \square

We can weaken the property in another way: suppose $A \subseteq M$. Then an undefinable, inductive set X is *A-Dolich* if whenever $a, b \in A$, then $a \in$

$\text{dcl}^{(\mathcal{M}, X)}(b)$ if and only if $a \in \text{dcl}(b)$. If A is finite, then we can use a resplendency argument to obtain an A -Dolich set.

Proposition 6. *Let $\mathcal{M} \models \text{PA}$ be countable and recursively saturated, and suppose $A \subseteq M$ is finite. Then there is an A -Dolich set $X \subseteq M$.*

Proof. This argument is due to Roman Kossak (via personal communication). Let $A = \{a_0, \dots, a_n\}$. If, for each i and j , $a_i \in \text{dcl}(a_j)$, then any undefinable, inductive X will work. Suppose that for some i and j , $a_i \notin \text{dcl}(a_j)$. Let T be the theory containing $\text{Th}(\mathcal{M}, a_0, \dots, a_n)$, all sentences of the form $t(a_j) \neq a_i$ whenever $a_i \notin \text{dcl}(a_j)$ where t is a term in the expanded language (with a predicate for X), and the following statements asserting X is undefinable and inductive:

$$\forall c \exists x \neg(\phi(x, c) \leftrightarrow x \in X) : \phi(x, y) \in \mathcal{L}_{\text{PA}}$$

$$\forall c (\phi(0, c) \wedge \forall x \phi(x, c) \rightarrow \phi(x + 1, c)) \rightarrow \forall x \phi(x, c) : \phi(x, y) \in \mathcal{L}_{\text{PA}}(X)$$

To see that this theory is consistent, suppose $a_i \notin \text{dcl}(a_j)$, and suppose $\phi_0(x, y), \dots, \phi_{k-1}(x, y)$ are formulas with two free variables. We will show that we can find a definable set X such that for each $i < k$ and $c \in M$, X and the set defined by $\phi_i(x, c)$ differ.

In fact, we only need to ensure that, for each formula $\phi(x, y)$, there is some X which is not defined by $\phi(\cdot, c)$ for any $c \in M$. That is, we can assume $k = 1$. This is because if $\phi_0(x, y), \dots, \phi_{k-1}(x, y)$ are any formulas, let $\phi(x, y)$ be the

formula

$$\exists z \bigvee_{i=0}^{k-1} (y = kz + i \implies \phi(x, z))$$

Then, for each $i < k$, and $c \in M$,

$$\mathcal{M} \models \forall x \phi_i(x, c) \leftrightarrow \phi(x, kc + i)$$

So we let $\phi(x, y)$ be some \mathcal{L}_{PA} formula. Let $\psi(x)$ be the following formula:

$$\exists z (x = 2z \wedge \neg \phi(2z, z)) \vee (x = 2z + 1 \wedge \phi(2z, z))$$

If X is the set defined by $\psi(x)$, X must be unbounded, since either $2c$ or $2c+1$ is in X for each $c \in M$. Further, $2c \in X$ if and only if $\mathcal{M} \models \neg \phi(2c, c)$ so X differs from the set defined by $\phi(\cdot, c)$ for each $c \in M$. Since X is definable and $a_i \notin \text{dcl}(a_j)$, then $a_i \notin \text{dcl}^{(\mathcal{M}, X)}(a_j)$ and hence T is consistent.

By resplendence \mathcal{M} has an expansion $(\mathcal{M}, X) \models \text{T}$, and by definition, $a_i \notin \text{dcl}^{(\mathcal{M}, X)}(a_j)$. \square

We can also show the following:

Proposition 7. *Let $\mathcal{M} \models \text{PA}$ be countable and recursively saturated, and suppose $A \subseteq M$ is finite. Then there is an inductive, undefinable $X \subseteq M$ such that for all $a, b \in A$, $\text{tp}(a) = \text{tp}(b)$ if and only if $\text{tp}^{(\mathcal{M}, X)}(a) = \text{tp}^{(\mathcal{M}, X)}(b)$.*

Proof. The proof of this result is very similar to the previous proof. Let $A = \{a_0, \dots, a_n\}$, and let T be the theory containing $\text{Th}(\mathcal{M}, a_0, \dots, a_n)$, all

sentences of the form $\forall x(\phi(a, x) \leftrightarrow \phi(b, x))$, where $\phi(u, x)$ is a formula in the expanded language $\mathcal{L}_{\text{PA}} \cup \{X\}$ and $a, b \in A$ are such that $\text{tp}(a) = \text{tp}(b)$, as well as the statements asserting X is undefinable and inductive. The same argument as in the previous proof shows that this theory is consistent and therefore by resplendency \mathcal{M} has an expansion $(\mathcal{M}, X) \models T$. \square

The following definitions are needed to state the next result. This result is also due to Roman Kossak (via personal communication).

Definition 4. A set $A \subseteq M$ is *small* if there is $c \in M$ such that $A = \{(c)_n : n \in \omega\}$.

Definition 5 ([KP77]). Let $I \subseteq_{\text{end}} M$. We say I is a *strong cut* in \mathcal{M} if for each $a \in M$, there is $c > I$ such that for all $i \in I$, $(a)_i > I$ if and only if $(a)_i > c$.

Proposition 8. *Let $\mathcal{M} \models \text{PA}$ be countable and recursively saturated and suppose ω is a strong cut in \mathcal{M} . Then if $A \subseteq M$ is small, there is an undefinable, inductive A -Dolich set X .*

Proof. Let $A = \{(c)_n : n \in \omega\}$. First we show that the set $\{\langle i, j \rangle : i, j \in \omega, (c)_i \in \text{Scl}((c)_j)\}$ is coded in \mathcal{M} using the fact that ω is strong.

Let $\langle t_i(x) : i \in \omega \rangle$ be a recursive enumeration of all Skolem terms. Then let $f(i, j)$ be the least n such that $t_n((c)_i) = (c)_j$. If S is an inductive satisfaction class for \mathcal{M} , then f is definable in (\mathcal{M}, S) . Therefore the following statement

is true in (\mathcal{M}, S) :

$$\forall k \exists x \forall i < k \forall j < k (x)_{\langle i, j \rangle} = f(i, j)$$

Let $k > \omega$ be any non-standard element of \mathcal{M} and let a be such that $\forall i < k \forall j < k (a)_{\langle i, j \rangle} = f(i, j)$. Then there is $b > \omega$ such that for all $n \in \omega$, $(a)_n \in \omega$ if and only if $(a)_n < b$.

Let T be the theory including $\text{Th}(\mathcal{M}, a, b, c)$, all the statements indicating that X is inductive and undefinable, and all statements of the form $\{t((c)_i) \neq (c)_j : (a)_{\langle i, j \rangle} > b\}$, where t is a Skolem term in the language $\mathcal{L}_{\text{PA}} \cup \{X\}$. A similar argument as the proof of Proposition 6 shows that T is consistent and therefore, by resplendency, \mathcal{M} has an expansion $(\mathcal{M}, X, a, b, c) \models T$ and X is A -Dolich. \square

2.5 Diversity in Elementary Substructures

The following definition is due to Schmerl [Sch04].

Definition 6. Let $\mathcal{M} \models \text{PA}$. If no two elementary substructures of \mathcal{M} are isomorphic, then \mathcal{M} is called *diverse*.

Ehrenfeucht's Lemma ([Ehr73]) states that if $t(a) = b$ for some Skolem term t , then $\text{tp}(a) \neq \text{tp}(b)$. This has a simple consequence:

Lemma 9. *If $\text{Lt}(\mathcal{M})$ is a linear order, then \mathcal{M} is diverse.*

Proof. Suppose $\mathcal{K}_1 \neq \mathcal{K}_2 \in \text{Lt}(\mathcal{M})$. Since $\text{Lt}(\mathcal{M})$ is a linear order, we can assume $\mathcal{K}_1 \prec \mathcal{K}_2$ and there is some $a \in K_2 \setminus K_1$. Since $\text{Scl}(a) \not\prec \mathcal{K}_1$, $\mathcal{K}_1 \prec \text{Scl}(a)$. Therefore, for each $b \in K_1$, there is some Skolem term t such that $\mathcal{M} \models t(a) = b$. Therefore $\text{tp}(a) \neq \text{tp}(b)$ for each $b \in K_1$ and there can be no isomorphism between \mathcal{K}_1 and \mathcal{K}_2 . \square

In particular, we get the following result:

Corollary 2. *There are 2^{\aleph_0} non-isomorphic diverse Enayat models.*

In fact we have more. We have showed that for each $\alpha < \omega_1$, there is an Enayat model whose substructure lattice is isomorphic to $(\alpha, <)$. Further, there is an Enayat model whose substructure lattice is isomorphic to $1 + \omega^*$. These Enayat models are also all diverse. Non-diverse Enayat models also exist.

Corollary 3. *Given any finite distributive lattice D that is not a chain, there is a non-diverse Enayat model \mathcal{M} whose substructure lattice is isomorphic to $D \oplus \mathbf{2}$.*

Proof. Theorem 2 of [Sch08] implies that for any countable, non-standard model $\mathcal{M} \models \text{PA}$, and any finite distributive lattice D , \mathcal{M} has a cofinal non-diverse extension \mathcal{N} . Let $T \neq \text{TA}$ be a completion of PA and \mathcal{M}_T a prime model of T . There is a non-diverse $\mathcal{N} \models \text{PA}$ that is a cofinal extension of \mathcal{M}_T . Let \mathcal{M} be a conservative, superminimal extension of \mathcal{N} . By Theorem 2.2.13 in [KS06], \mathcal{M} is Enayat. \square

In fact, this shows that, for each finite distributive lattice D that is not a chain, there are 2^{\aleph_0} non-isomorphic non-diverse \mathcal{M} whose substructure lattices are all isomorphic to $D \oplus \mathbf{2}$.

2.6 Open Problems

From the work in this section, we can find many Enayat models that have the substructure lattice $1 + \omega^*$ simply by taking a superminimal conservative extension of the \mathcal{M} provided by either Proposition 2 or Theorem 1.

Corollary 4. *For every completion T of PA, there are 2^{\aleph_0} non-isomorphic Enayat models of T whose substructure lattice is isomorphic to $1 + \omega^*$.*

□

The following is still open:

Problem 1. *Classify all Enayat models according to better-known model-theoretic properties. In particular, determine all infinite lattices which can be realized as the substructure lattice of an Enayat model.*

Proposition 2 answers the question about whether there are Enayat models whose substructure lattices are linear orders but not well-orders. Our construction appears to answer a special case of a more general question about substructure lattices. As mentioned previously, our construction in Theorem 1 is not the first construction of a model whose substructure lattice is isomorphic to $1 + \omega^*$. Mills [Mil79] gave a construction showing that if D is an \aleph_1 -algebraic

distributive lattice, then there is $\mathcal{M} \models \text{PA}$ with $\text{Lt}(\mathcal{M}) \cong D$. Mills's construction, applied to the lattice $1 + \omega^*$, gives an end-extensional type producing $1 + \omega^*$. Because the type is end-extensional, it implies that the rankset of the ensuing ranked lattice would be $\{\mathbf{0}, \mathbf{1}\}$. Our construction gives a model \mathcal{M} such that $\text{Lt}(\mathcal{M}) = 1 + \omega^*$ and the rank function is the identity function. If a model \mathcal{M} is conservative over all its submodels, then the rank function on $\text{Lt}(\mathcal{M})$ must be the identity. Our construction leads to the following, more general question about ranked lattices:

Problem 2. *Let L be a lattice and r a rank function on L . Is there $\mathcal{M} \models \text{PA}$ with $\text{Ltr}(\mathcal{M}) = (L, r)$ such that whenever $\mathcal{K}_1 \prec \mathcal{K}_2$ are in the rankset of $\text{Lt}(\mathcal{M})$, then $\mathcal{K}_1 \prec_{\text{cons}} \mathcal{K}_2$?*

In other words, given $\mathcal{M} \models \text{PA}$ with $\text{Ltr}(\mathcal{M}) = (L, r)$, can we find $\mathcal{N} \models \text{PA}$ with $\text{Ltr}(\mathcal{N}) = (L, r)$ and, for each $\mathcal{K}_1 \prec_{\text{end}} \mathcal{K}_2 \prec \mathcal{N}$, the extension $\mathcal{K}_1 \prec \mathcal{K}_2$ is conservative? To ensure that \mathcal{N} is Enayat, we must also insist that it has no proper cofinal submodels. That means that $r(x) < \mathbf{1}$ whenever $x < \mathbf{1}$. If we additionally ensure that the rankset is well-ordered then such an \mathcal{N} will be Enayat by Proposition 1.

Schmerl's proof (Proposition 4) settles, negatively, the question raised by Dolich. We can weaken the question in a number of ways:

Problem 3. *Let \mathcal{M} be a countable recursively saturated model of PA.*

1. *Is there an undefinable class X of \mathcal{M} such that the definable closure relation in (\mathcal{M}, X) coincides with the definable closure relation in \mathcal{M} ?*

2. Given $n < \omega$, is there an undefinable inductive set X such that for all $a, b \in M$, a is definable by a Σ_n formula from b in (\mathcal{M}, X) if and only if a is definable by a Σ_n formula from b in \mathcal{M} ?

Note that if X is a class of \mathcal{M} , then $(\mathcal{M}, X) \models \text{I}\Sigma_0$. If the answer to Problem 3.1 is positive, then we can ask whether there is an undefinable X such that $(\mathcal{M}, X) \models \text{I}\Sigma_n$ and does not change the dcl relation.

Propositions 6 and 8 show that for countable, recursively saturated models \mathcal{M} , there are subsets $A \subseteq M$ such that A -Dolich sets exist. We do not have a complete characterization of which subsets A are such that A -Dolich sets exist.

Problem 4. *Let \mathcal{M} be a countable, recursively saturated model. Characterize those $A \subseteq M$ that are such that A -Dolich sets exist.*

Proposition 7 also raises a similar question: what are those $A \subseteq M$ such that there are inductive, undefinable $X \subseteq M$ where, for each $a, b \in A$, $\text{tp}(a) = \text{tp}(b)$ if and only if $\text{tp}^{(\mathcal{M}, X)}(a) = \text{tp}^{(\mathcal{M}, X)}(b)$?

Chapter 3

Coded Sets and Interstructure

Lattices

3.1 Introduction

In [KP92], it is shown that for countable models, every set which can be coded in some elementary end extension can be coded in a minimal elementary end extension. Schmerl [Sch14] characterizes the collection of subsets of countable models of PA that can be coded in minimal elementary end extensions. This result is in some way a generalization of the work in [KP92]: it shows that if \mathcal{M} is a countable model of PA and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ is such that there is any finitely generated elementary end extension \mathcal{N} such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$, then there is a minimal elementary end extension \mathcal{N}' such that $\text{Cod}(\mathcal{N}'/\mathcal{M}) = \mathfrak{X}$. This work is expanded upon in [Sch15], where the collections of subsets of a

model of arbitrary cardinality which can be coded in a minimal elementary end extension are characterized. Minimal extensions are, of course, realizations of the interstructure lattice $\mathbf{2}$, so we can attempt to generalize this by the following two questions:

Question 2. *In the vein of [KP92], given a lattice L and a (countable) model \mathcal{M} , if $X \subseteq M$ can be coded in some elementary end extension, can it be coded in an elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?*

Question 3. *In the vein of [Sch14] and [Sch15], given a lattice L and a (countable) model \mathcal{M} , what collections $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ are such that there is an elementary end extension \mathcal{N} with $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$?*

We attempt to extend this work to further understand which subsets can be coded in elementary end extensions with more general interstructure lattices. As an example, we know that, if $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$, the pentagon lattice, then \mathcal{N} is not a conservative extension of \mathcal{M} .

3.2 Coding Sets in Elementary Extensions

The main theorem of [Sch15] states that if $\mathcal{M} \models \text{PA}$, $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ is countably generated, contains $\text{Def}(\mathcal{M})$, $(\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0^*$, and every set that is Π_1^0 -definable in $(\mathcal{M}, \mathfrak{X})$ is a countable union of sets that are Σ_1^0 -definable, then \mathcal{M} has a minimal elementary end extension \mathcal{N} such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. In particular, if $(\mathcal{M}, \mathfrak{X}) \models \text{ACA}_0$ and \mathfrak{X} is countably generated, then \mathcal{M} has

an elementary end extension such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. This next result is a corollary of [Sch15], showing that every inductive set can be coded in a minimal elementary end extension.

Corollary 5. *Suppose $\mathcal{M} \models \text{PA}$ and $X \subseteq M$ is inductive. Then \mathcal{M} has a minimal elementary end extension \mathcal{N} such that X is coded in \mathcal{N} .*

Proof. Since X is inductive, then $(\mathcal{M}, \text{Def}(\mathcal{M}, X)) \models \text{ACA}_0$. It is also clear that $\text{Def}(\mathcal{M}, X)$ is countably generated as the 0-definable sets in (\mathcal{M}, X) generate it. \square

[KP92] states that if \mathcal{M} is countable, then any class which can be coded in some elementary end extension can be coded in a minimal elementary end extension. We can extend this result to ω_1 -like models, or, more generally, κ -like models for regular, uncountable κ .

Corollary 6. *Let κ be a regular, uncountable cardinal and suppose \mathcal{M} is κ -like. If $X \subseteq M$ is a class of \mathcal{M} , then \mathcal{M} has a minimal elementary end extension coding X .*

Proof. Since every class of a κ -like model is inductive, this follows from Corollary 5. \square

We note that a κ -like model may be rather classless, in which case this result follows from Gaifman's general theorem on the existence of minimal elementary end extensions.

Next we generalize [Sch14, Theorem 5] in a different direction, showing that if D is a finite distributive lattice and $\mathcal{M} \models \text{PA}$ is countable, then there is $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$ if and only if there is a minimal elementary end extension \mathcal{N} such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. This result generalizes [KP92] and [Sch14, Theorem 5]. If D is any finite distributive lattice, then the sets which can be coded in an elementary end extension yielding the interstructure lattice D are exactly the same as the sets which can be coded in any finitely generated elementary end extension.

To prove this result, first we revisit the proof of [KS06, Theorem 4.3.7], which proves that, if D is a finite distributive lattice, then every countable model \mathcal{M} has a conservative elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$. By inspecting this proof, we obtain the following result:

Lemma 10. *Let D be a finite distributive lattice, $\mathcal{M} \models \text{PA}$ countable and suppose $\mathcal{M} \prec \mathcal{K}$ is a minimal extension. Then there is $\mathcal{K} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$ and $\text{Cod}(\mathcal{K}/\mathcal{M}) = \text{Cod}(\mathcal{N}/\mathcal{M})$.*

Proof. First we will review the proof of [KS06, Theorem 4.3.7], in order to show how to prove this Lemma.

If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ are models (of any theory), and $f_i : \mathcal{M}_0 \rightarrow \mathcal{M}_i$ (for $i = 1, 2$) are elementary embeddings, then an *amalgamation* of \mathcal{M}_1 and \mathcal{M}_2 over \mathcal{M}_0 is a model \mathcal{M}_3 with elementary embeddings $g_i : \mathcal{M}_i \rightarrow \mathcal{M}_3$ (for $i = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$. In our arguments, these embeddings will all be the identities, so we will have $\mathcal{M}_0 \prec \mathcal{M}_1, \mathcal{M}_2 \prec \mathcal{M}_3$, and $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$.

In general, models can have many non-isomorphic amalgamations. We can restrict this to situations in which \mathcal{M}_3 is generated (in some sense) over \mathcal{M}_0 by $M_1 \cup M_2$, but even then there may still be non-isomorphic amalgamations in the case of models of PA. We will describe a case in which we can ensure that amalgamations would be unique (up to isomorphism). This definition comes from [KS06, Definition 2.3.7].

Given a model $\mathcal{M} \models \text{PA}$, the language $\mathcal{L}_{\text{PA}}(M)$ is the expansion of \mathcal{L}_{PA} by adding constant symbols for each element of \mathcal{M} . If $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$, then for each $\mathcal{L}(N)$ -formula $\phi(x)$, there is a $\mathcal{L}(M)$ -formula $\sigma_\phi(x)$ such that for each $a \in M$, $\mathcal{N} \models \phi(a)$ iff $\mathcal{M} \models \sigma_\phi(a)$.

Definition 7. Suppose $\mathcal{M} \prec \mathcal{M}_1$ and $\mathcal{M} \prec_{\text{cons}} \mathcal{M}_2$. For each $\mathcal{L}_{\text{PA}}(M_2)$ formula $\phi(x)$, let $\sigma_\phi(x)$ be the $\mathcal{L}_{\text{PA}}(M)$ -formula such that for each $a \in M$, $\mathcal{M}_2 \models \phi(a)$ if and only if $\mathcal{M} \models \sigma_\phi(a)$. Then the *principal amalgamation* $\mathcal{M}_3 = \mathcal{M}_1 * \mathcal{M}_2$ is the prime model of the theory T containing the elementary diagram of \mathcal{M} along with all $\mathcal{L}_{\text{PA}}(M_1 \cup M_2)$ -sentences of the form $\phi(b)$, where $\phi(x)$ is an $\mathcal{L}(M_2)$ formula, $b \in M_1$ and $\mathcal{M}_1 \models \sigma_\phi(b)$.

In this case, Theorem 2.3.2 of [KS06] shows that the principal amalgamation is unique (up to isomorphism), and is a conservative extension of \mathcal{M}_1 .

Definition 8. Let L be any lattice. The lattice $L \times \mathbf{2}$ is the lattice $\{\langle a, i \rangle : a \in L, i \in \mathbf{2}\}$, ordered so that $\langle a, i \rangle \leq \langle b, j \rangle$ if and only if $a \leq b$ and $i \leq j$. If $x \in L$, the x -doubling extension of L is the lattice $\{\langle r, i \rangle \in L \times \mathbf{2} : i = 0 \vee r \geq x\}$.

By identifying $a \in L$ with the element $\langle a, 0 \rangle$, it is clear that, for any $x \in L$,

the x -doubling extension is an extension of L .

If L is the substructure lattice of some \mathcal{M} , then for each $a \in L$, there is some \mathcal{N} such that $\text{Lt}(\mathcal{N})$ is isomorphic to the a -doubling extension of L . This result is [KS06, Theorem 4.3.2], which states that if $\mathcal{M}_0 \prec \mathcal{M}$ are countable, then there is a superminimal, conservative extension $\mathcal{N}_0 \succ \mathcal{M}_0$ such that $\text{Lt}(\mathcal{M} * \mathcal{N}_0)$ is the \mathcal{M}_0 -doubling extension of $\text{Lt}(\mathcal{M})$. This result generalizes to interstructure lattices: if $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \mathcal{M} \prec \mathcal{N}$, and \mathcal{N} is such that $\text{Lt}(\mathcal{N})$ is the \mathcal{M}_1 -doubling extension of $\text{Lt}(\mathcal{M})$, then $\text{Lt}(\mathcal{N}/\mathcal{M}_0)$ is the \mathcal{M}_1 -doubling extension of $\text{Lt}(\mathcal{M}/\mathcal{M}_0)$.

Let D be a finite distributive lattice. An element $a \in D$ is called join-irreducible if whenever $x \vee y = a$, then either $x = a$ or $y = a$. Birkhoff's Representation Theorem ([Bir67]) states that, letting J_0 be the set of non-zero join-irreducible elements of D , then the map $x \mapsto \{a \in J_0 : a \leq x\}$ is a lattice isomorphism. A corollary of this representation theorem is [KS06, Theorem 4.3.6]: if D is a finite distributive lattice, with non-zero join-irreducible elements a_1, \dots, a_n ordered so that if $a_i < a_j$, then $i < j$, then there is a sequence of lattices L_1, \dots, L_n such that $L_1 = \mathbf{2}$, L_{i+1} is the $(a_{i+1} \wedge \bigvee L_i)$ -doubling extension of L_i , and $L_n \cong D$.

Using this sequence, we start with $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_1 = \mathcal{K}$, and then we obtain a sequence $\mathcal{M}_2, \dots, \mathcal{M}_n$ such that $\text{Lt}(\mathcal{M}_i/\mathcal{M}) \cong L_i$ by ensuring that we take the correct doubling extension at each stage. In this way, we get that $\mathcal{M}_i \prec_{\text{cons}} \mathcal{M}_{i+1}$ when $i \geq 1$. Let $\mathcal{N} = \mathcal{M}_n$, and then $\text{Cod}(\mathcal{N}/\mathcal{M}) = \text{Cod}(\mathcal{K}/\mathcal{M})$. \square

Theorem 2. *Suppose \mathcal{M} is countable, $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, and D is a finite distribu-*

tive lattice. The following are equivalent:

- (1) \mathfrak{X} is countable, $\text{Def}(\mathcal{M}) \subseteq \mathfrak{X}$ and $(\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0^*$.
- (2) There is a minimal elementary end extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.
- (3) There is an elementary end extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \text{D}$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

Proof. (1) \iff (2) is the content of [Sch14, Theorem 5]. If $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ is any finitely generated extension (in particular if the lattice $\text{Lt}(\mathcal{N}/\mathcal{M})$ is finite), then $\text{Cod}(\mathcal{N}/\mathcal{M})$ must be countable, and, by [Sch14, Proposition 1], $(\mathcal{M}, \text{Cod}(\mathcal{N}/\mathcal{M})) \models \text{WKL}_0^*$. As mentioned before, $\text{Cod}(\mathcal{N}/\mathcal{M})$ will also always contain the definable sets if $\mathcal{M} \prec_{\text{end}} \mathcal{N}$, so (3) \implies (1). So it remains to show (2) \implies (3).

Let \mathcal{M} , D , and \mathfrak{X} be as in the statement of the theorem. Then by [Sch14, Theorem 5], there is a minimal elementary end extension \mathcal{K} of \mathcal{M} such that $\text{Cod}(\mathcal{K}/\mathcal{M}) = \mathfrak{X}$. By Lemma 10, there is $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \text{Cod}(\mathcal{K}/\mathcal{M})$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \text{D}$. \square

Corollary 7. *Suppose $\mathcal{M} \models \text{PA}$ is countable and D is a finite distributive lattice. If $X \subseteq M$ can be coded in some elementary end extension, then \mathcal{M} has an elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \text{D}$ and $X \in \text{Cod}(\mathcal{N}/\mathcal{M})$.*

In the proof of Lemma 10, we make use of Theorem 4.3.2 in [KS06], showing that if $\mathcal{M}_0 \prec \mathcal{M}$ are countable models, then \mathcal{M}_0 has a superminimal conser-

vative elementary extension \mathcal{N}_0 such that if $\mathcal{N} = \mathcal{M} * \mathcal{N}_0$, then $\text{Lt}(\mathcal{N})$ is the \mathcal{M}_0 -doubling extension of $\text{Lt}(\mathcal{M})$. Uncountable models do not have superminimal extensions, so generalizing this result would require us to weaken superminimality. However, superminimality is necessary in this proof. If $\mathcal{M}_1 \prec \mathcal{M}$, and $\mathcal{M}_1 \prec \mathcal{N}$ is such that $\text{Lt}(\mathcal{M} * \mathcal{N})$ is the \mathcal{M}_1 -doubling extension of $\text{Lt}(\mathcal{M})$, then \mathcal{N} must be a superminimal extension of \mathcal{M}_1 , and therefore \mathcal{M}_1 must be countable. To see this, suppose \mathcal{N} is not a superminimal extension of \mathcal{M}_1 , and let $b \in N \setminus M_1$ be such that $\text{Scl}(b) \neq \mathcal{N}$. $\text{Scl}(b) \in \text{Lt}(\mathcal{M} * \mathcal{N}) \setminus \text{Lt}(\mathcal{M})$. Therefore, $\mathcal{M}_1 \prec \text{Scl}(b)$ as $\text{Lt}(\mathcal{M} * \mathcal{N})$ is the \mathcal{M}_1 -doubling extension of $\text{Lt}(\mathcal{M})$. However, $\text{Scl}(b) \cap M = N \cap M = M_1$. If L is any finite lattice and L' is the e -doubling extension of L , we can see that whenever $\langle r, 1 \rangle \neq \langle s, 1 \rangle \in L'$, then $\langle r, 1 \rangle \wedge \langle \mathbf{1}, 0 \rangle \neq \langle s, 1 \rangle \wedge \langle \mathbf{1}, 0 \rangle$, where $\mathbf{1}$ is the top element of L .

If we only care about interstructure lattices, however, then we do not need the extension to be superminimal. We need a different property, which can help us generalize this result to uncountable models. Before we state the following definition, recall that if $\mathcal{M} \prec \mathcal{N}$ and $a \in N \setminus M$, then $\mathcal{M}(a) = \text{Scl}^{\mathcal{N}}(M \cup \{a\})$.

Definition 9. Let $\mathcal{M}_0 \prec \mathcal{M} \prec \mathcal{N}$ be models of PA. \mathcal{N} is a *superminimal extension of \mathcal{M} relative to \mathcal{M}_0* if whenever $b \in N \setminus M$, then $\mathcal{M}_0(b) = \mathcal{N}$.

Obviously, \mathcal{N} is a superminimal extension of \mathcal{M} relative to \mathcal{M} if and only if it is a minimal extension of \mathcal{M} . In general, \mathcal{N} is a superminimal extension of \mathcal{M} relative to \mathcal{M}_0 if and only if $\text{Lt}(\mathcal{N}/\mathcal{M}_0)$ is the same as $\text{Lt}(\mathcal{M}/\mathcal{M}_0)$ with a new element added on top; we write this using the “lattice sum” notation:

$\text{Lt}(\mathcal{N}/\mathcal{M}_0) \cong \text{Lt}(\mathcal{M}/\mathcal{M}_0) \oplus \mathbf{2}$.

To understand the following results, we must review minimal types. Minimal types were introduced by Gaifman in [Gai76]. A type $p(x)$ is *minimal* if it is an unbounded complete type and whenever $\mathcal{M} \prec \mathcal{M}(a)$, where a realizes $p(x)$ and $a > M$, then $\mathcal{M}(a)$ is a minimal extension of \mathcal{M} .

Lemma 11. *If $\mathcal{M}_0 \prec \mathcal{M}$ is a finitely generated extension, then there is \mathcal{N} a conservative, superminimal extension of \mathcal{M} relative to \mathcal{M}_0 .*

Proof. Suppose $\mathcal{M} = \mathcal{M}_0(a)$. We show that there is a minimal extension $\mathcal{M}(c)$ such that $\text{tp}(c)$ is rare and $a \in \text{Scl}(c)$. A type $p(x)$ over a complete theory T is called *rare* if, whenever $t(x)$ is a Skolem term, then there is a formula $\phi(x) \in p(x)$ such that

$$T \vdash \forall x \forall y [\phi(x) \wedge \phi(y) \wedge x < y \rightarrow t(x) < y]$$

An equivalent definition comes from studying gaps of elements: given a model $\mathcal{M} \models \text{PA}$, the gap of an element a , denoted $\text{gap}(a)$, is the set of all b such that there are Skolem terms f and g such that $\mathcal{M} \models a < f(b) \wedge b < g(a)$. [KS06, Theorem 3.1.16] characterizes rare types as those types $p(x)$ such that for any model \mathcal{M} and $c \in M$ realizing $p(x)$, if $b \in \text{gap}(c)$ then $c \in \text{Scl}(b)$. Therefore if $\mathcal{M}(c)$ is a minimal extension of \mathcal{M} , $\text{tp}(c)$ is rare, and $a \in \text{Scl}(c)$, then, for any $b \in \mathcal{M}(c)$, either $b \in \mathcal{M}$ or $\mathcal{M}_0(b) = \mathcal{M}(c)$. Hence, $\mathcal{M}(c)$ is a superminimal extension of \mathcal{M} relative to \mathcal{M}_0 .

To get an extension $\mathcal{M}(c)$ with $\text{tp}(c)$ rare and $a \in \text{Scl}(c)$, we apply [Sch02,

Lemma 2.1], which states that if $\mathcal{M} \models \text{PA}$, $\phi(u, x)$ is any formula, and $a \in M$ is such that $\phi(a, x)$ defines an unbounded subset of M , then \mathcal{M} has a minimal, conservative extension \mathcal{N} in which there is $c \in N$ realizing a rare type and $\mathcal{N} \models \phi(a, c)$. Let $\phi(u, x)$ be the formula $\exists v x = \langle u, v \rangle$ and apply this Lemma. Then $\mathcal{M}(c)$ is a minimal extension of \mathcal{M} , with $a \in \text{Scl}(c)$, and $\text{tp}(c)$ is rare, and therefore $\mathcal{M}(c)$ is a superminimal extension of \mathcal{M} relative to \mathcal{M}_0 . \square

Corollary 8. *Suppose $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ is a minimal extension. Let $n \geq 2 \in \omega$. There is $\mathcal{N} \prec_{\text{cons}} \mathcal{N}_1$ such that $\text{Lt}(\mathcal{N}_1/\mathcal{M}) \cong \mathbf{n}$.*

Proof. For $n = 2$, the hypothesis that there is a minimal elementary end extension satisfies this condition. The inductive step appeals to the previous lemma. \square

Notice that in this result, we “double” the top element of the lattice $\mathbf{2}$ repeatedly. We can also double the bottom element.

Lemma 12. *Let $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ and suppose $\text{Lt}(\mathcal{N}/\mathcal{M})$ is finite. Then \mathcal{N} has a conservative, minimal elementary end extension \mathcal{N}_1 such that $\text{Lt}(\mathcal{N}_1/\mathcal{M})$ is the 0-doubling extension of $\text{Lt}(\mathcal{N}/\mathcal{M})$.*

Proof. Let $p(x)$ be a minimal type over \mathcal{M} and let \mathcal{M}_1 be an elementary extension of \mathcal{M} generated by an element realizing $p(x)$. We claim that if $\mathcal{N}_1 = \mathcal{N} * \mathcal{M}_1$, then $\text{Lt}(\mathcal{N}_1/\mathcal{M})$ is the 0-doubling extension of $\text{Lt}(\mathcal{N}/\mathcal{M})$.

Let $a \in \mathcal{M}_1$ be an element realizing $p(x)$. Then $\mathcal{M}_1 = \mathcal{M}(a)$. Given $\mathcal{K} \in \text{Lt}(\mathcal{N}/\mathcal{M})$, clearly $\mathcal{K} * \mathcal{M}(a) \in \text{Lt}(\mathcal{N}_1/\mathcal{M})$. Further, if $\mathcal{K} \in \text{Lt}(\mathcal{N}_1/\mathcal{M})$

and $\mathcal{K} \not\prec \mathcal{N}$, then we show that $a \in K$. Let $b \in K \setminus N$, and suppose $m \in N$ and $t(u, x)$ are such that $\mathcal{N}_1 \models t(m, a) = b$. Because $p(x)$ is a minimal type, there are Skolem terms t_1 and t_2 and a formula $\phi(x) \in p(x)$ such that the following statement holds in \mathcal{M} ([KS06, Theorem 3.2.13]), and, by elementarity, in \mathcal{N} :

$$\forall u \exists w [\forall x > w \phi(x) \rightarrow t(u, x) = t_1(u)] \vee [\forall x > w \phi(x) \rightarrow t_2(t(u, x)) = x] \quad (3.1)$$

In particular, there is some $w \in N$ such that

$$\forall x > w (\phi(x) \rightarrow t(m, x) = t_1(m)) \vee \forall x > w (\phi(x) \rightarrow t_2(t(m, x)) = x)$$

holds in \mathcal{N} . Since $b \notin N$, it cannot be the case that $\mathcal{N} \models \forall x > w \phi(x) \rightarrow t(m, x) = t_1(m)$. Therefore $\mathcal{N}_1 \models t_2(b) = a$, so $a \in K$.

We let L be the 0-doubling extension of $\text{Lt}(\mathcal{N}/\mathcal{M})$ and define $\alpha : L \rightarrow \text{Lt}(\mathcal{N}_1/\mathcal{M})$ as follows: $\alpha(\langle \mathcal{K}, 0 \rangle) = \mathcal{K}$ and $\alpha(\langle \mathcal{K}, 1 \rangle) = \mathcal{K} * \mathcal{M}(a)$. We claim this is an isomorphism of lattices. If $\mathcal{K}_1 \preceq \mathcal{K}_2$, and $i \leq j \in \{0, 1\}$, then $\alpha(\langle \mathcal{K}_1, i \rangle) \preceq \alpha(\langle \mathcal{K}_2, j \rangle)$. Suppose $\alpha(\langle \mathcal{K}_1, i \rangle) \preceq \alpha(\langle \mathcal{K}_2, j \rangle)$. It is impossible for $i = 1$ and $j = 0$ since $a \in \alpha(\langle \mathcal{K}_1, 1 \rangle) \setminus \alpha(\langle \mathcal{K}_2, 0 \rangle)$, so $i \leq j$. We show that $\mathcal{K}_1 \preceq \mathcal{K}_2$. If $i = j = 0$, then this is clear. If not, then $j = 1$. Suppose $b \in K_1$. Then there are $m \in K_2$ and a Skolem term t such that $\mathcal{K}_2 * \mathcal{M}(a) \models t(m, a) = b$. There are terms $t_1(u)$ and $t_2(y)$ and a formula $\phi(x) \in p(x)$ such that the statement (3.1) holds in \mathcal{M} and, by elementarity, in $\mathcal{K}_2 * \mathcal{M}(a)$. Since $a \notin K_1$, $\mathcal{K}_2 * \mathcal{M}(a) \models t_2(b) \neq a$. Therefore, $\mathcal{K}_2 * \mathcal{M}(a) \models t_1(m) = b$, so $b \in K_2$.

Lastly, we show that if $\mathcal{K} \in \text{Lt}(\mathcal{N}_1/\mathcal{M})$, then there is some $\mathcal{K}_0 \in \text{Lt}(\mathcal{N}/\mathcal{M})$ and $i \in \{0, 1\}$ such that $\alpha(\langle \mathcal{K}_0, i \rangle) = \mathcal{K}$. If $\mathcal{K} \in \text{Lt}(\mathcal{N}/\mathcal{M})$, then this is clear by letting $\mathcal{K}_0 = \mathcal{K}$ and $i = 0$. Otherwise, let $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{N}$ and $i = 1$. Then we must show $\mathcal{K}_0 * \mathcal{M}(a) = \mathcal{K}$. We already showed that if $\mathcal{K} \in \text{Lt}(\mathcal{N}_1/\mathcal{M}) \setminus \text{Lt}(\mathcal{N}/\mathcal{M})$, then $a \in K$. Let $b \in K$. There are $m \in N$ and t such that $\mathcal{N}_1 \models t(m, a) = b$. Let $g(x, y)$ be the Skolem term defined as the least u such that $t(u, x) = y$, and let $g(a, b) = c$. Then $c \in K$ and $\mathcal{K} \models t(c, a) = b$. We claim that $c \in K_0$. We must show that $c \in N$, but since $\mathcal{N}_1 \models t(m, a) = b$ with $m \in N$, $\mathcal{N}_1 \models g(a, b) \leq m$. Since $\mathcal{N} \prec_{\text{end}} \mathcal{N}_1$, this implies that $c \in N$ and therefore $c \in K_0$. \square

Corollary 9. *Let $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ be a minimal extension and $n \geq 2$. There is \mathcal{N}_1 such that $\mathcal{M} \prec_{\text{end}} \mathcal{N}_1$, $\text{Cod}(\mathcal{N}_1/\mathcal{M}) = \text{Cod}(\mathcal{N}/\mathcal{M})$ and $\text{Lt}(\mathcal{N}_1/\mathcal{M}) \cong \mathbf{B}_n$.*

Proof. If $n = 2$, we apply the previous lemma to $\mathcal{M} \prec \mathcal{N}$ and obtain \mathcal{N}_1 such that $\text{Lt}(\mathcal{N}_1/\mathcal{M}) \cong \mathbf{B}_2$. Noticing that \mathbf{B}_{n+1} is the 0-doubling extension of \mathbf{B}_n , the more general result follows by induction. \square

To summarize the results in this section, we first recall some definitions from [Sch14] and [Sch15]. Let $\mathcal{M} \models \text{PA}$ and $\mathcal{M} \prec \mathcal{N}$. We say \mathcal{N} is *countably generated* over \mathcal{M} if there is a countable set $A \subseteq N$ such that $\mathcal{N} = \mathcal{M}(A)$. That is, \mathcal{N} is the smallest elementary extension of \mathcal{M} containing A .

Theorem 3. *Let $\mathcal{M} \models \text{PA}$ and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$. The following are equivalent:*

- (1) *There is a countably generated $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$,*

and every set that is Π_1^0 -definable in $(\mathcal{M}, \mathfrak{X})$ is the union of countably many Σ_1^0 -definable sets.

(2) There is a minimal elementary end extension $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

(3) If $n \geq 2$, then there is an elementary end extension $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{n}$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

(4) If \mathbf{B} is a finite Boolean Algebra, then there is an elementary end extension $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

Proof. (1) \iff (2) is the main result of [Sch15]. (3) \implies (1) and (4) \implies (1) are a result of Corollary 2.2 in [Sch15], which states that if $\text{Lt}(\mathcal{N}/\mathcal{M})$ is countable and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$, then every set that is Π_1^0 -definable in $(\mathcal{M}, \mathfrak{X})$ is the union of countably many Σ_1^0 -definable sets.

(2) \implies (3) is Corollary 8 and (2) \implies (4) is Corollary 9. \square

3.3 Open Problems

For finite distributive lattices, the general case for models of arbitrary cardinality remains open. We list a conjecture here:

Conjecture 1. *Let $\mathcal{M} \models \text{PA}$ and \mathbf{D} be a finite distributive lattice. Suppose $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$. The following are equivalent:*

(1) There is a minimal elementary end extension $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

- (2) *There is an elementary end extension $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{D}$ and $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.*

Kossak and Paris [KP92] showed that for countable models \mathcal{M} , if X can be coded in any elementary end extension, then X can be coded in a minimal extension. The corresponding statement for uncountable models is open. Schmerl [Sch15] does not quite answer this question for us as we do not know if, given a set X which can be coded in some elementary end extension, we can find an \mathfrak{X} satisfying the hypotheses of the main theorem of [Sch15] such that $X \in \mathfrak{X}$.

Problem 5. *Suppose $\mathcal{M} \models \text{PA}$ and $X \subseteq M$ can be coded in some elementary end extension. Can X be coded in a minimal elementary end extension?*

The most significant open problem in this area is to understand what is happening in the case of the pentagon lattice \mathbf{N}_5 . We know that if $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$, then \mathcal{N} is not a conservative extension of \mathcal{M} . That is, $\text{Cod}(\mathcal{N}/\mathcal{M}) \neq \text{Def}(\mathcal{M})$. Nothing else is known at this point.

Problem 6. *Given $\mathcal{M} \models \text{PA}$, characterize the sets $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ (in terms of the second order properties of $(\mathcal{M}, \mathfrak{X})$) which can arise as $\text{Cod}(\mathcal{N}/\mathcal{M})$ for some $\mathcal{N} \succ_{end} \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.*

In the study of the relationship between interstructure lattices and the second-order properties of the sets coded in the extension, we can attempt to find a theory strong enough that would ensure that \mathbf{N}_5 can be realized

in an extension coding a prescribed collection of subsets. One such “strong enough” theory would be $\Pi_1^1\text{-CA}_0$, the strongest of the “Big Five” subsystems of second-order arithmetic studied in reverse mathematics.

Problem 7. *Suppose $\mathcal{M} \models \text{PA}$, $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ is countably generated, and $(\mathcal{M}, \mathfrak{X}) \models \Pi_1^1\text{-CA}_0$. Is there $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$?*

As there are some models that only have conservative end extensions, not every model can have an elementary end extension whose interstructure lattice is \mathbf{N}_5 . Any countable model has such an elementary end extension, but it is unknown if there are any uncountable models which do.

Problem 8. *Characterize the models $\mathcal{M} \models \text{PA}$ which have an elementary end extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.*

The lattice problem for models of PA is open. That is, there are finite lattices L such that we do not know if there is $\mathcal{M} \models \text{PA}$ with $\text{Lt}(\mathcal{M}) \cong L$. In general, if \mathcal{M}_0 is a minimal non-standard model, and there is some $\mathcal{M} \succ \mathcal{M}_0$ such that $\text{Lt}(\mathcal{M}) \cong L$, then there is $\mathcal{M} \succ_{\text{cof}} \mathcal{M}_0$ with $\text{Lt}(\mathcal{M}) \cong L$. There are some lattices which can only be realized as substructure lattices of models \mathcal{M} which are cofinal extensions of their minimal submodels. We do not have a complete characterization of when a lattice can be realized as an end extension and when a substructure lattice can only be realized as a cofinal extension. Ultimately, we hope that the study of the relationship between lattices and coded sets will give us a better understanding of these problems:

Problem 9. *Classify those L that can be realized as a substructure lattice of a model of PA, and classify those L which can be the substructure lattice of an elementary end extension of a prime model of a completion of PA.*

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