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Some Metric Properties of the Teichmüller Space of a Closed Set in the Riemann Sphere

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Abstract

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Advisors: Yunping Jiang and Sudeb Mitra

Associated to each closed subset E of the Riemann sphere $\widehat{\mathbb{C}}$, there is a contractible complex Banach manifold with a basepoint; this was first studied by G. Lieb in his Cornell University doctoral dissertation [15]. We call this the *Teichmüller space of the closed set E, denoted by $T(E)$*. Throughout this thesis, the blanket assumption will be that E is a closed set in $\widehat{\mathbb{C}}$ and that $0, 1, \infty$ belong to E. The Teichmüller space $T(E)$ is intimately related with holomorphic motions of the closed set E.

In this thesis we study several metric and analytic properties of $T(E)$.

For the Teichmüller space of a Riemann surface, the principle of Teichmüller contraction was introduced by Gardiner in the paper [9]. In [18] Mitra proved a δ-ϵ form of Teichmüller contraction for the generalized Teichmüller space $T(E)$. Our first theorem in this thesis, is to extend Earle's form of Teichmüller contraction

to the space $T(E)$. This is **Theorem A** of this thesis. It improves and sharpens Mitra's δ-ϵ form of Teichmüller contraction for $T(E)$.

We then study holomorphic isometries for $T(E)$, thereby generalizing Theorem 5 of the paper of Earle, Kra and Krushkaĺ [7]. This is **Theorem B** of this thesis.

In his paper [6], Earle proved a version of Schwarz's lemma for Teichmüller space of a Riemann surface. **Theorem C** of this thesis extends Earle's result to the generalized Teichmüller space $T(E)$.

Finally, we study complex geodesics and unique extremality for $T(E)$, in order to generalize Theorem 6 of the paper of Earle-Kra-Krushkaĺ [7]. This is **Theorem D** of this thesis.

When E is finite there is a natural identification of $T(E)$ with the classical Teichmüller space $Teich(\widehat{\mathbb{C}} \setminus E)$. For the more general case, when E is infinite, we consider an increasing sequence $\{E_n\}$ of finite subsets of E, such that $0, 1, \infty$ always belong to E_n and $\bigcup_n E_n$ is dense in E.

It was proved by Mitra in [17] that $T(E)$ is the metric inverse limit of the pointed metric spaces $\{T(E_n)\}$. A basic technique in this thesis is to exploit this approximation. We summarize the main results of this thesis as follows. The precise definitions are given in Sections/Chapters indicated below. In what follows, let Δ be the open unit disk in the complex plane, that is, $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Theorem A (Teichmüller contraction in $T(E)$)**.** *Let $\mu \in M(\mathbb{C})$, $P_E(\mu) = \tau \in T(E)$ and μ_0 be extremal in the E-equivalence class of μ; that is $P_E(\mu_0) = P_E(\mu)$ and*

$\|\mu_0\|_\infty \leq \|\mu\|_\infty$. *Let* $k = \|\mu\|_\infty$ *and* $k_0 = \|\mu_0\|_\infty$, *also*

$$K = \frac{1+k}{1-k} \qquad \text{and} \qquad K_0 = \frac{1+k_0}{1-k_0},$$

We define ℓ_μ *as in §1.6*

$$\ell_\mu(\phi) = \iint_{\mathbb{C}} \mu\phi dxdy \quad, \forall\phi \in A(E), \quad \forall\mu \in L^\infty(\mathbb{C}).$$

and

$$\|\ell_\mu\|_{T(E)} = \sup\{|\ell_\mu(\phi)|, \phi \in A(E), \|\phi\|_1 = 1\}.$$

Then

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq K - K_0.$$

Theorem B (Holomorphic isometries from Δ to $T(E)$)**.** *Let* $f : \Delta \to T(E)$ *be holomorphic, and let* $t_1 \in \Delta$, *if*

1. $d_{T(E)}(f(t_1), f(t_2)) = \rho_\Delta(t_1, t_2)$ *for some* $t_2 \in \Delta \setminus \{t_1\}$ *or*

2. $\|f'(t_1)\|_{T(E)} = F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1-|t_1|^2}$

then f *is a holomorphic isometry.*

[The definitions of $d_{T(E)}$ and $F_{T(E)}$ are given in §1.6. The definition of holomorphic isometry is given in Chapter 3.]

Theorem C (Schwarz's lemma for $T(E)$)**.** *Let* $f : \Delta \to T(E)$ *be holomorphic and* $t \in \Delta \setminus \{0\}$ *with* $f(0) = 0$. *If either of the inequalities*

1. $\|f'(0)\|_{T(E)} \leq 1$

2. $k_0(t) \leq |t|$

is strict, then both are strict and

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \|f^{'}(0)\|_{T(E)}\Big) \leq 2\rho_\Delta(0,t).$$

[The definition of ρ_Δ is given in §1.3.]

Theorem D (Complex geodesics in $T(E)$)**.** *Let* $\mu_0 \in M(\mathbb{C})$*,* $\mu_0 \neq 0$ *and* μ_0 *be extremal in its* E-equivalence *class. Then the following four statements are equivalent:*

1. *The Beltrami coefficient* μ_0 *is uniquely extremal and* $|\mu_0| = \|\mu_0\|_\infty$ *a.e.*

2. *There exists only one geodesic segment joining* $P_E(0)$ *and* $P_E(\mu_0)$*.*

3. *There exists only one holomorphic isometry* $f : \Delta \to T(E)$ *such that* $f(0) = P_E(0)$ *and* $f(\|\mu_0\|_\infty) = P_E(\mu_0)$*.*

4. *There exists only one holomorphic map* $g : \Delta \to M(\mathbb{C})$ *such that* $g(0) = 0$ *and* $P_E(g(\|\mu_0\|_\infty)) = P_E(\mu_0)$*.*

[The definitions of complex geodesics and unique extremality are given in Chapter 5.]

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Dedicated to my uncle
Professor Kshitish Chattopadhyay.

Contents

Chapter 1

Introduction

1.1 Quasiconformal mappings

Throughout this dissertation we will use $\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and Δ for the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.1. A complex-valued function $w = f(z)$ defined in a region $V \subset \mathbb{C}$ is called a *quasiconformal mapping* if it is a sense-preserving homeomorphism of V onto its image and its complex distributional derivatives

$$w_z = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \text{ and } w_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

are Lebesgue measurable locally square integrable functions on V that satisfy the inequality $|w_{\bar{z}}| \leq k|w_z|$ almost everywhere in V, for some real number k with $0 \leq k < 1$.

If $w = f(z)$ is a quasiconformal mapping defined on the region V then the function w_z is known to be nonzero almost everywhere on V. Therefore the function

$$\mu_f = \frac{w_{\bar{z}}}{w_z}$$

is well defined L^∞ function on V, called the *complex dilatation* or the *Beltrami coefficient* of f. The L^∞ norm of every Beltrami coefficient is less than one.

The positive number

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the *dilatation* of f. We say that f is K-*quasiconformal* if f is a quasiconformal mapping and $K(f) \leq K$.

We call a homeomorphism of $\widehat{\mathbb{C}}$ *normalized* if it fixes the points 0, 1, and ∞.

We will always denote by $M(\mathbb{C})$ the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Then, for each μ in $M(\mathbb{C})$, there exists a unique normalized quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself that has Beltrami coefficient μ (see [1]); this quasiconformal map will be denoted by w^μ. Furthermore, we have the following fundamental theorem due to Ahlfors and Bers: for every fixed $z \in \mathbb{C}$, the map $\mu \mapsto w^\mu(z)$ of $M(\mathbb{C})$ into $\mathbb{C}$ is holomorphic (see [2]).

1.2 Infinite Dimensional Holomorphy

Teichmüller spaces are complex manifolds modeled on finite or infinite dimensional complex Banach spaces. In this section we will discuss some basic facts about holomorphic maps between Banach spaces. The standard reference is [4].

Definition 1.2. Let E and F be two complex Banach spaces of finite or infinite dimensions. Let U be a nonempty open subset in E. A mapping $f : U \to F$ is *holomorphic* if and only if it is locally bounded and the *complex Gâteaux derivative* $f'(x)(\lambda)$ at $x \in U$ in the direction $\lambda \in E$ defined as:

$$f'(x)(\lambda) = \lim_{t \to 0} \frac{f(x + t\lambda) - f(x)}{t} \in F$$

exists (in the norm of F) for $t \in \mathbb{C}$ and for each $(x, \lambda) \in U \times E$.

If f is holomorphic, then $f'(x)(\lambda) : E \to F$ is a continuous $\mathbb{C}$-linear map for each $x \in U$. This map is called the *Fréchet derivative* of f at x. We note the following

Proposition 1.3. *A map $f : U \to F$ is holomorphic if and only if for each x in U there is a continuous complex linear map $f'(x) : E \to F$ such that*

$$\frac{||f(x + y) - f(x) - f'(x)(y)||_F}{||y||_E} \to 0 \tag{1.2.1}$$

as $y \to 0$ in E.

Remark 1.4. The map $f : U \to F$ has a real Fréchet derivative at each point $x \in U$. If $f'(x) : E \to F$ is a continuous, real linear map that satisfies (1.2.1)

Definition 1.5. A *complex Banach manifold modeled on a complex Banach space* E is a topological (Hausdorff) space X with an open covering U_i such that:

(i) For each U_i there is a homeomorphism $\varphi : U_i \to \varphi_i(U_i) \subset E$, where $\varphi_i(U_i)$ is an open subset of E, and

(ii) The transition homeomorphism

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a holomorphic isomorphism, for each pair of indices i, j.

The following concept will be crucial throughout this dissertation.

Definition 1.6. A holomorphic map f between two complex Banach manifolds X and Y is a *holomorphic split submersion* if it is an open mapping and has holomorphic local sections. This means for any given x in X there is a holomorphic right inverse to f defined in some neighborhood of $f(x)$ and mapping $f(x)$ to x.

1.3 The Kobayashi metric

Recall that the Poincaré metric on Δ is given by

$$\rho_\Delta(z_1, z_2) = \tanh^{-1} \left\| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right\|$$

for z_1 and z_2 in Δ.

Let X be a connected complex manifold and let $\mathcal{O}(\Delta, X)$ be the set of holomorphic maps from Δ into X. The Kobayashi function $\delta_X : X \times X \to [0, \infty]$ is defined for all x and y in X by

$$\delta_X(x, y) = \inf\{\rho_\Delta(0, t) : f(0) = x \text{ and } f(t) = y \text{ for some } f \in \mathcal{O}(\Delta, X)\} \quad (1.3.1)$$

provided the set of maps described above is nonempty, and $+\infty$ otherwise. It easily follows from this definition that if X and Y are connected complex Banach manifolds and $f : X \to Y$ is a holomorphic map, then

$$\delta_Y(f(x_1), f(x_2)) \leq \delta_X(x_1, x_2) \text{ for all } x_1, x_2 \in X.$$

Definition 1.7. The Kobayashi (pseudo)metric ρ_X is defined to be the largest (pseudo)metric on X such that

$$\rho_X(x, y) \leq \delta_X(x, y) \text{ for all } x, y \in X.$$

If δ_X is a metric, then ρ_X and δ_X are equal. In any case, if X and Y are connected complex manifolds and $f : X \to Y$ is a holomorphic map, we have

$$\rho_Y(f(x_1), f(x_2)) \leq \rho_X(x_1, x_2) \text{ for all } x_1, x_2 \in X. \tag{1.3.2}$$

Equality holds in Equation (1.3.2) if f is biholomorphic.

Recall the open unit ball $M(\mathbb{C})$ of the complex Banach space $L^\infty(\mathbb{C})$.

Proposition 1.8. *The Kobayashi metric on $M(\mathbb{C})$ is given by:*

$$\rho_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\mu}\nu} \right\|_\infty$$

for all μ, ν in $M(\mathbb{C})$. The infinitesimal Kobayashi metric on $M(\mathbb{C})$ is given by:

$$K_M(\mu, \lambda) = \left\| \frac{\lambda}{1 - |\mu|^2} \right\|_\infty$$

for μ in $M(\mathbb{C})$ and λ in $L^\infty(\mathbb{C})$.

See Proposition 7.25 in [8].

We need the following inequality of Lindelöf; see [16] and also [11]. For the reader's convenience, we give a self-contained proof.

Before the proof however, we proceed with a few preliminaries. To begin, for $z, a \in \Delta$ define the **pseudo-hyperbolic metric on** Δ to be

$$d_\Delta(z,a) = |\phi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right|.$$

Observe that if $f : \Delta \to \Delta$ is holomorphic then by the Schwarz-Pick Theorem

$$d_\Delta(f(z), f(a)) = \left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\bar{a}z}\right| = d_\Delta(z,a)$$

and if $f \in \mathrm{Aut}(\Delta)$ then equality holds, i.e.

$$d_\Delta(f(z), f(a)) = d_\Delta(z,a) \qquad \text{for all } z, a \in \Delta.$$

Lemma 1.9. *d_Δ is a metric*

Proof. Let $z, a \in \Delta$. For ease of notation, we will omit the subscript Δ and use d to denote d_Δ. That $d(z,a) \geq 0$ is clear, as is the fact that $d(z,a) = 0$ if and only if $z = a$.

To see that $d(z,a) = d(a,z)$ simply observe that

$$d(z,a) = |\phi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| \qquad \text{and} \qquad d(a,z) = |\phi_z(a)| = \left|\frac{a-z}{1-\bar{z}a}\right|. \qquad (1.3.3)$$

But

$$
\begin{aligned}
|1-\bar{a}z|^2 &= (1-\bar{a}z)(1-a\bar{z}) \\
&= (1-\bar{z}a)(1-z\bar{a}) \\
&= |1-\bar{z}a|^2
\end{aligned}
$$

and so $|1-\bar{a}z| = |1-\bar{z}a|$. This implies, referring back to (1.3.3), that $d(z,a) = d(a,z)$.

It remains to show the triangle inequality, and we claim it suffices to prove

$$d(t_1,t_2) \le |t_1| + |t_2|, \quad \text{for all } t_1,t_2 \in \Delta. \tag{1.3.4}$$

Indeed for $z,w,a \in \Delta$, the triangle inequality $d(z,w) \le d(z,a) + d(a,w)$ holds if and only if $d(\phi_a(z),\phi_a(w)) \le d(\phi_a(z),0) + d(0,\phi_a(w))$ since $\phi_a \in \mathrm{Aut}(\Delta)$ and $\phi_a(a) = 0$. Letting $t_1 = \phi_a(z)$ and $t_2 = \phi_a(w)$, this becomes $d(t_1,t_2) \le d(t_1,0) + d(0,t_2)$ which is precisely (1.3.4) since $d(z,0) = |z|$ for any $z \in \Delta$.

Thus our goal is to prove (1.3.4), but in fact we aim to prove a much stronger result, namely:

Lemma 1.10. *For any $t_1,t_2 \in \Delta$,*

$$d(t_1,t_2) \le \frac{|t_1|+|t_2|}{1+|t_1||t_2|.}$$

This of course implies (1.3.4) since $1/(1+|t_1||t_2|) \le 1$. To prove Lemma 1.10, we begin by observing that

$$
\begin{aligned}
1 - d(t_1, t_2)^2 &= 1 - \left| \frac{t_1 - t_2}{1 - t_1 \bar{t}_2} \right|^2 \\
&= \frac{(1 - t_1 \bar{t}_2)(1 - \bar{t}_1 t_2) - (t_1 - t_2)(\bar{t}_1 - \bar{t}_2)}{|1 - t_1 \bar{t}_2|^2} \\
&= \frac{(1 - |t_1|^2)(1 - |t_2|^2)}{|1 - t_1 \bar{t}_2|^2} \\
&\geq \frac{(1 - |t_1|^2)(1 - |t_2|^2)}{(1 + |t_1||t_2|)^2}, \qquad (1.3.5)
\end{aligned}
$$

where the last line follows from the triangle inequality. We also compute the following:

$$
\begin{aligned}
d(|t_1|, -|t_2|) &= \left| \frac{|t_1| - (-|t_2|)}{1 - |t_1|(\overline{-|t_2|})} \right| \\
&= \left| \frac{|t_1| + |t_2|}{1 + |t_1||t_2|} \right| \\
&= \frac{|t_1| + |t_2|}{1 + |t_1||t_2|}
\end{aligned}
$$

and so

$$
1 - d(|t_1|, -|t_2|)^2 = 1 - \left(\frac{|t_1| + |t_2|}{1 + |t_1||t_2|} \right)^2 = \frac{(1 - |t_1|^2)(1 - |t_2|^2)}{(1 + |t_1||t_2|)^2}.
$$

Comparing the previous line with (1.3.5) we see that

$$
1 - d(t_1, t_2)^2 \geq 1 - d(|t_1|, -|t_2|)^2.
$$

Which implies

$$d(t_1, t_2) \leq d(|t_1|, -|t_2|) = \frac{|t_1| + |t_2|}{1 + |t_1||t_2|}.$$

This proves Lemma 1.10 and hence Lemma 1.9 as well. □

Our next goal is to prove Lindelöf's Inequality as stated previously, but first we derive one more inequality concerning the metric d.

Claim: *For all $z, a \in \Delta$, the pseudo-hyperbolic metric d satisfies*

$$d(|z|, |a|) \leq d(z, a) \leq d(|z|, -|a|).$$

Proof. We need only prove the first inequality since the second was verified in the proof of Lemma 1.10. To this end, note that

$$d(|z|, |a|) = \left| \frac{|z| - |a|}{1 - |z||a|} \right| = \frac{||z| - |a||}{1 - |z||a|}$$

and so

$$\begin{aligned} 1 - d(|z|, |a|)^2 &= 1 - \frac{||z| - |a||^2}{(1 - |z||a|)^2} \\ &= 1 - \frac{(|z| - |a|)(|\bar{z}| - |\bar{a}|)}{(1 - |z||a|)^2} \\ &= 1 - \frac{(|z| - |a|)(|z| - |a|)}{(1 - |z||a|)^2} \\ &= \frac{(1 - |z|^2)(1 - |a|^2)}{(1 - |z||a|)^2}. \end{aligned} \tag{1.3.6}$$

But the above is less than or equal to $1 - d(z, a)^2$. Indeed, consider the following:

$$1 - d(z,a)^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2} \leq \frac{(1-|z|^2)(1-|a|^2)}{(1-|z||a|)^2} = 1 - d(|z|,|a|)^2.$$

The leftmost equality was derived in the proof of Lemma 1.10, and the inequality follows since $|1-\bar{a}z| \geq ||1| - |a\bar{z}|| = |1-|a||z||$. Of course the rightmost equality is (1.3.6). From this we conclude $d(|z|,|a|) \leq d(z,a)$ as desired.

□

Finally we are ready to derive Lindelöf's inequality.

Proposition 1.11 (Lindelöf)**.** *If* $f : \Delta \to \Delta$ *is holomorphic then*

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|} \quad \textit{for all } z \in \Delta.$$

Proof. We prove the rightmost inequality first. Let $z \in \Delta$. By our claim above we know that

$$d(|f(z)|, |f(0)|) \ \leq \ d(f(z), f(0)) \ \leq \ d(|z|, 0).$$

Which by definition of d implies

$$\left| \frac{|f(z)| - |f(0)|}{1 - |f(z)||f(0)|} \right| \leq |z| \tag{1.3.7}$$

and so $||f(z)| - |f(0)|| \leq |z|(1 - |f(z)||f(0)|)$. Thus

$$\begin{aligned} |f(z)| &\leq ||f(z)| - |f(0|| + |f(0)| \\ &\leq |z|(1 - |f(z)||f(0)|) + |f(0)|. \end{aligned}$$

Rearranging terms we find that $|f(z)|(1+|z||f(0)|) \leq |f(0)|+|z|$ and so

$$|f(z)| \leq \frac{|f(0)|+|z|}{1+|f(0)||z|}$$

as desired. For the lower bound on $|f(z)|$ observe that (1.3.7) implies

$$\begin{aligned}|f(0)| &\leq ||f(0)|-|f(z)|| + |f(z)| \\ &\leq |z|(1-|f(z)||f(0)|) + |f(z)|\end{aligned}$$

and hence, again with some rearranging, $|f(0)|+|z||f(z)||f(0)| \leq |f(z)|+|z|$ and thus

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \leq |f(z)|.$$

□

The following theorem appears in Earle's paper [6] as Theorem 3. This result will be crucial in our thesis. For the reader's convenience, we give a complete proof.

Theorem 1.12 (Earle [6]). *Let V be a complex Banach space and $g : \Delta \to V$ be a holomorphic map with $g(0) = 0$ and $\|g(t)\| \leq 1$, $\forall t \in \Delta$. Fix $t \in \Delta \setminus \{0\}$. If either of the inequalities*

1. $\|g'(0)\| \leq 1$
2. $\|g(t)\| \leq |t|$

is strict then both are strict and

$$\rho_\Delta\Big(\frac{\|g(t)\|}{|t|}, \|g'(0)\|\Big) \leq \rho_\Delta(0,t).$$

Proof. We observe that Lindelöf's inequality is valid with the assumption $f : \Delta \to \overline{\Delta}$, since if for any $t_0 \in \Delta$, $|f(t_0)| = 1$, by maximum modulus principle $f(t) = f(t_0), \forall t \in \Delta$ and the inequality reduces to

$$\frac{1 - |t|}{1 - |t|} \leq |f(t)| \leq \frac{1 + |t|}{1 + |t|},$$

which implies $|f(t)| = 1$ and hence the inequality is trivially true.

So, without loss of generality we can say that if f is holomorphic on Δ and $|f(t)| \leq 1$ for all $t \in \Delta$ then $\forall t \in \Delta$ the folowing inequality is true

$$\frac{|f(0)| - |t|}{1 - |t||f(0)|} \leq |f(t)| \leq \frac{|f(0)| + |t|}{1 + |t||f(0)|}. \tag{1.3.8}$$

Let $\ell : V \to \mathbb{C}$ be a linear functional on V with $\|\ell\| = 1$, where

$$\|\ell\| = \sup\left\{\frac{|\ell(x)|}{\|x\|}, \quad x \in V, \|x\| \neq 0\} = \sup\{|\ell(x)|, \quad x \in V, \|x\| = 1\right\}.$$

Consider the holomorphic map $g : \Delta \to V$, we already know that $\|g(t)\| \leq 1$ and hence from the definition of $\|\ell\|$ we conclude that $|\ell(g(t))| \leq 1$, So the function $h : \Delta \to \overline{\Delta}$ defined as $h(t) = \ell(g(t))$ is holomorphic, since, g is holomorphic and ℓ is linear. Moreover $h(0) = 0$, so applying Schwarz's lemma we see that $|h(t)| = |\ell(g(t))| \leq |t|$. Hence $\frac{|\ell(g(t))|}{|t|} \leq 1, \forall t \in \Delta$

Also since ℓ is linear, we see that $h^{'}(0) = \ell(g^{'}(0))$ and hence again by Schwarz's lemma and the fact $\|\ell\| = 1$ we see that $|h^{'}(0)| = |\ell(g^{'}(0)| \leq 1$.

Now consider the function f defined as follows

$$f(t) = \begin{cases} \frac{\ell(g(t))}{t} & \text{if } t \neq 0 \\ \ell(g^{'}(0)) & \text{if } t = 0. \end{cases}$$

From our discussion above we see that $|f(t)| \leq 1$ and since

$$f(0) = \ell(g^{'}(0)) = \lim_{t \to 0} \frac{\ell(g(t))}{t},$$

f is holomorphic.

So we see that $f : \Delta \to \overline{\Delta}$ is holomorphic and hence we get

$$\frac{|f(0)| - |t|}{1 - |t||f(0)|} \leq |f(t)| \leq \frac{|f(0)| + |t|}{1 + |t||f(0)|}, \quad \forall t \in \Delta.$$

Hence, since our only assumption on ℓ was $\|\ell\| = 1$, we conclude $\forall \ell : V \to \mathbb{C}$, linear, with $\|\ell\| = 1$ the following is true for all $t \in \Delta$,

$$\frac{|\ell(g^{'}(0))| - |t|}{1 - |t||\ell(g^{'}(0))|} \leq \left| \frac{\ell(g(t))}{t} \right| \leq \frac{|\ell(g^{'}(0))| + |t|}{1 + |t||\ell(g^{'}(0))|}. \tag{1.3.9}$$

By Hahn Banach theorem we can choose ℓ_1 and ℓ_2, linear functionals on V with $\|\ell_1\| = \|\ell_2\| = 1$ such that $\ell_1(g^{'}(0)) = \|g^{'}(0)\|$ and $\ell_2(g(t)) = \|g(t)\|$.

Since (1.3.9) is valid for all ℓ, linear functionals with $\|\ell\| = 1$, it's true for ℓ_1 and ℓ_2 as well.

Notice first $|\ell_1(g(t))| \leq \|g(t)\|$ since $\|\ell_1\| = 1$ and hence $\frac{|\ell_1(g(t))|}{|t|} \leq \frac{\|g(t)\|}{|t|}$, $\forall t \in \Delta \setminus \{0\}$.

Also, observe

$$\frac{|\ell_2(g^{'}(0))| + |t|}{1 + |t||\ell_2(g^{'}(0))|} \leq \frac{\|g^{'}(0)\| + |t|}{1 + |t|\|g^{'}(0)\|} \tag{1.3.10}$$

$$\Leftrightarrow (|\ell_2(g^{'}(0))| + |t|)(1 + |t|\|g^{'}(0)\|)$$

$$\leq (\|g^{'}(0)\| + |t|)(1 + |t||\ell_2(g^{'}(0))|)$$

$$\Leftrightarrow |\ell_2(g^{'}(0))| + |t||\ell_2(g^{'}(0))|\|g^{'}(0)\| + |t| + |t|^2\|g^{'}(0)\|$$

$$\leq \|g^{'}(0)\| + |t||\ell_2(g^{'}(0))|\|g^{'}(0)\| + |t| + |t|^2|\ell_2(g^{'}(0))|$$

$$\Leftrightarrow |\ell_2(g^{'}(0))| + |t|^2\|g^{'}(0)\| \leq \|g^{'}(0)\| + |t|^2|\ell_2(g^{'}(0))|$$

$$\Leftrightarrow |\ell_2(g^{'}(0))|(1 - |t|^2) \leq \|g^{'}(0)\|(1 - |t|^2)$$

$$\Leftrightarrow |\ell_2(g^{'}(0))| \leq \|g^{'}(0)\|.$$

Since $\|\ell_2\| = 1$ we already know $|\ell_2(g^{'}(0))| \leq \|g^{'}(0)\|$ and so we conclude (1.3.10) is true as well.

Now fix a t in $\Delta \setminus \{0\}$.

Applying (1.3.9) on ℓ_1 we obtain

$$\frac{\|g^{'}(0)\| - |t|}{1 - |t|\|g^{'}(0)\|} = \frac{|\ell_1(g^{'}(0))| - |t|}{1 - |t||\ell_1(g^{'}(0))|} \leq \frac{|\ell_1(g(t))|}{|t|} \leq \frac{\|g(t)\|}{|t|}. \tag{1.3.11}$$

Applying (1.3.9) and (1.3.10) on ℓ_2 we obtain

$$\frac{\|g(t)\|}{|t|} = \frac{|\ell_2(g(t))|}{|t|} \leq \frac{|\ell_2(g^{'}(0))| + |t|}{1 + |t||\ell_2(g^{'}(0))|} \leq \frac{\|g^{'}(0)\| + |t|}{1 + |t|\|g^{'}(0)\|}. \tag{1.3.12}$$

Combining (1.3.11) and (1.3.12), we get

$$\frac{\|g'(0)\| - |t|}{1 - |t|\|g'(0)\|} \leq \frac{\|g(t)\|}{|t|} \leq \frac{\|g'(0)\| + |t|}{1 + |t|\|g'(0)\|}. \tag{1.3.13}$$

Now if $\|g'(0)\| = 1$, then left of (1.3.13) tells us

$$1 = \frac{1 - |t|}{1 - |t|} \leq \frac{\|g(t)\|}{|t|}.$$

Hence $\|g(t)\| \geq |t|$, so contrapositively $\|g(t)\| < |t| \Rightarrow \|g'(0)\| < 1$ and if $\|g(t)\| \geq |t|$ then of (1.3.13) gives us

$$\frac{\|g'(0)\| + |t|}{1 + |t|\|g'(0)\|} \geq 1$$

$$\Leftrightarrow \|g'(0)\| + |t| \geq 1 + |t|\|g'(0)\|$$

$$\Leftrightarrow \|g'(0)\|(1 - |t|) \geq (1 - |t|)$$

$$\Leftrightarrow \|g'(0)\| \geq 1.$$

Hence contrapositively $\|g'(0)\| < 1 \Rightarrow \|g(t)\| < |t|$. Hence we get

$$\|g'(0)\| < 1 \Leftrightarrow \|g(t)\| < |t|.$$

For a fixed t, from the left of (1.3.13) we have

$$\frac{\|g'(0)\| - |t|}{1 - |t|\|g'(0)\|} \leq \frac{\|g(t)\|}{|t|}$$

$$\Leftrightarrow \|g'(0)\| - |t| \leq \frac{\|g(t)\|}{|t|} - \frac{\|g(t)\|}{|t|}|t|\|g'(0)\|$$

$$\Leftrightarrow -|t|(1 - \frac{\|g(t)\|}{|t|}\|g^{'}(0)\|) \leq \frac{\|g(t)\|}{|t|} - \|g^{'}(0)\|$$

$$\Leftrightarrow -|t| \leq \frac{\frac{\|g(t)\|}{|t|} - \|g^{'}(0)\|}{1 - \frac{\|g(t)\|}{|t|}\|g^{'}(0)\|}. \tag{1.3.14}$$

From the right hand side of (1.3.13) we have

$$\frac{\|g(t)\|}{|t|} \leq \frac{\|g^{'}(0)\| + |t|}{1 + |t|\|g^{'}(0)\|}$$

$$\Leftrightarrow \frac{\|g(t)\|}{|t|} + \frac{\|g(t)\|}{|t|}|t|\|g^{'}(0)\| \leq \|g^{'}(0)\| + |t|$$

$$\Leftrightarrow \frac{\|g(t)\|}{|t|} - \|g^{'}(0)\| \leq |t|(1 - \frac{\|g(t)\|}{|t|}\|g^{'}(0)\|)$$

$$\Leftrightarrow \frac{\frac{\|g(t)\|}{|t|} - \|g^{'}(0)\|}{1 - \frac{\|g(t)\|}{|t|}\|g^{'}(0)\|} \leq |t|. \tag{1.3.15}$$

Combining (1.3.14) and (1.3.15) we get

$$\left| \frac{\frac{\|g(t)\|}{|t|} - \|g^{'}(0)\|}{1 - \frac{\|g(t)\|}{|t|}\|g^{'}(0)\|} \right| \leq |t|. \tag{1.3.16}$$

Hence from (1.3.16) we get

$$\tanh \rho_\Delta\Big(\frac{\|g(t)\|}{|t|}, \|g^{'}(0)\|\Big) \leq \tanh \rho_\Delta(0, t). \tag{1.3.17}$$

Since as tanh is a monotonically increasing bijection we finally get

$$\rho_\Delta\Big(\frac{\|g(t)\|}{|t|}, \|g^{'}(0)\|\Big) \leq \rho_\Delta(0, t).$$

□

1.4 Teichmüller space of a plane region

Let Ω be a plane region whose complement $\mathbb{C} \setminus \Omega$ contains at least two points. Two quasiconformal mappings f and g with domain Ω are said to be in the same Teichmüller class if and only if there is a conformal map h of $f(\Omega)$ onto $g(\Omega)$ such that the self-mapping $g^{-1} \circ h \circ f$ of Ω is isotopic to the identity rel the boundary of Ω. (This means that $g^{-1} \circ h \circ \ f$ extends to a homeomorphism of the closure of Ω onto itself that is isotopic to the identity by an isotopy that fixes the boundary pointwise.) The *Teichmüller space* $Teich(\Omega)$ is the set of all Teichmüller classes of quasiconformal mappings with domain Ω.

The Teichmüller class of f depends only on its Beltrami coefficient, which is a function μ in the open unit ball $M(\Omega)$ of the complex Banach space $L^\infty(\Omega)$. The canonical projection $\Phi : M(\Omega) \to Teich(\Omega)$ maps μ to the Teichmüller class of any quasiconformal map whose domain is Ω and whose Beltrami coefficient is μ. The basepoints of $M(\Omega)$ and $T(\Omega)$ are 0 and $\Phi(0)$ respectively.

Teichmüller metric. The *Teichmüller metric* on $M(\Omega)$ is defined as:

$$\rho_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \overline{\mu}\nu} \right\|_\infty \tag{1.4.1}$$

for any μ and ν in $M(\Omega)$. This Teichmüller metric ρ_M provides the same topology to $M(\Omega)$ as does the Banach norm topology of $L^\infty(\Omega)$. The *Teichmüller metric* on $Teich(\Omega)$ is the quotient metric:

$$d_T(s,t) = \inf\{\rho_M(\mu, \nu) : \mu \text{ and } \nu \text{ in } M(\Omega), \Phi(\mu) = s, \text{ and } \Phi(\nu) = t\} \tag{1.4.2}$$

for all s and t in $Teich(\Omega)$. The infimum is always attained (see [1]) and we can choose μ in $\Phi^{-1}(s)$ arbitrarily so that we have

$$d_T(\Phi(\mu), t) = \min\{d_M(\mu, \nu) : \nu \in M(\Omega) \text{ and } \Phi(\nu) = t\}$$

for all μ in $M(\Omega)$ and t in $Teich(\Omega)$. It follows easily that the canonical projection Φ is an open continuous map with respect to the Teichmüller metric.

A fundamental fact in the theory of Teichmüller spaces is the following:

Theorem 1.13. *There is a unique complex Banach manifold structure on* $Teich(\Omega)$ *such that* $\Phi : M(\Omega) \to Teich(\Omega)$ *is a holomorphic split submersion, that is* Φ *is surjective, holomorphic, its derivative is surjective and* Φ *has local holomorphic sections.*

See, for example, [13] or [19].

The tangent space at the basepoint. We identify $L^\infty(\Omega)$ with the dual space of $L^1(\Omega)$ in the obvious way, by sending μ in $L^\infty(\Omega)$ to the bounded linear functional

$$\ell_\mu(f) = \iint_\Omega \mu(z) f(z)\, dx\, dy, \qquad f \in L^1(\Omega).$$

The space $A(\Omega)$ of integrable holomorphic functions on Ω is a closed subspace of $L^1(\Omega)$, and hence a (complex) Banach space. The orthogonal complement of $A(\Omega)$ is the set:

$$A(\Omega)^\perp = \{\mu \in L^\infty(\Omega) : \ell_\mu(\phi) = 0 \text{ for all } \phi \text{ in } A(\Omega)\}.$$

A fundamental result of Teichmüller theory is

Proposition 1.14 (Teichmüller's Lemma). *$ker(\Phi'(0)) = A(\Omega)^{\perp}$.*

For the proof we refer to [19].

By a well-known principle (see [20]), we can identify $L^{\infty}(\Omega)/A(\Omega)^{\perp}$ with the dual space of $A(\Omega)$. Hence Teichmüller's lemma implies the following

Corollary 1.15. *The tangent space to $Teich(\Omega)$ at its basepoint is naturally isomorphic to the Banach dual $A^*(\Omega)$ of $A(\Omega)$.*

Teichmüller contraction:

For each tangent vector v to $Teich(\Omega)$ at $\Phi(0)$, there exists some μ in $L^{\infty}(\Omega)$ such that $v = \Phi'(0)\mu$. The natural isomorphism sends v to the linear functional $\varphi \mapsto \ell_{\mu}(\varphi)$ on $A(\Omega)$.

We write

$$\|\ell_{\mu}\|_T = \sup_{\|\phi\|=1} \left\{ \left| \iint_{\mathbb{C}} \mu\phi dxdy \right|, \phi \in A(\Omega) \right\}$$

It is therefore clear that $\|\ell_{\mu}\|_T \leq \|\mu\|_{\infty}$.

Let μ_0 in $M(\Omega)$ be extremal in its Teichmüller class; which means $\Phi(\mu_0) = \Phi(\mu)$ and $\|\mu_0\|_{\infty} \leq \|\mu\|_{\infty}$. Let $k_0 = \|\mu\|_{\infty}$, and $k = \|\mu\|_{\infty}$; also, let

$$K_0 = \frac{1+k_0}{1-k_0} \qquad \text{and } K = \frac{1+k}{1-k}.$$

In [6], Earle proved the sharp form of the principle of Teichmüller contraction:

Theorem 1.16 (Earle). *Let μ be in $M(\Omega)$ with $\|\mu\|_{\infty} = k < 1$, and let K and K_0*

be defined as above. Then

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_T) \leq K - K_0.$$

Changing the basepoint. Let h be a quasiconformal map whose domain is Ω and whose image is a plane region $\widetilde{\Omega}$. Let g be any quasiconformal map whose domain is $\widetilde{\Omega}$. By definition, the *allowable map* h^* from $Teich(\widetilde{\Omega})$ to $Teich(\Omega)$ maps the Teichmüller class of g to the Teichmüller class of $g \circ h$.

Proposition 1.17. *The allowable map $h^* \colon Teich(\widetilde{\Omega}) \to Teich(\Omega)$ is biholomorphic. If μ is the Beltrami coefficient of h, then h^* maps the basepoint of $Teich(\widetilde{\Omega})$ to the point $\Phi(\mu)$ in $Teich(\Omega)$.*

See [19] for the details.

We conclude this section with the following important result due to Royden when Ω is a finite analytic type Riemann surface and Gardiner when Ω is an infnite analytic type Riemann surface. A simpler proof was given by Earle, Kra and Krushkaĺ in [7].

Proposition 1.18. *The Teichmüller metric on $Teich(\Omega)$ is the same as its Kobayashi metric.*

Proof. See [7]. □

1.5 Product Teichmüller space

Let $\{X_n\}_{n=1}^{\infty}$ be a countable collection of nonempty plane regions, none of which contains 0 or 1. Let X be the disjoint union $\coprod_n X_n$. We will define the Teichmüller space of X, which we denote by $T(X)$; we will also discuss some of its properties. For the details we refer the reader to [8] or [15].

Teichmüller space of X. For each $n \geq 1$ let $Teich(X_n)$ be the Teichmüller space of the region X_n, let 0_n be the basepoint of $Teich(X_n)$, and let d_{T_n} be the Teichmüller metric on $Teich(X_n)$. By definition, the Teichmüller space $Teich(X)$ is the set of sequences $t = \{t_n\}_{n=1}^{\infty}$ such that t_n belongs to $Teich(X_n)$ for each n and

$$\sup\{d_{T_n}(0_n, t_n) : n \geq 1\} < \infty.$$

The basepoint of $Teich(X)$ is the sequence $0 = \{0_n\}$ whose nth term is the basepoint of $Teich(X_n)$.

Teichmüller metric on $Teich(X)$. The Teichmüller metric on $Teich(X)$ is defined by:

$$d_T(s,t) = \sup\{d_{T_n}(s_n, t_n) : n \geq 1\}.$$

Let $L^{\infty}(X)$ be the complex Banach space of sequences $\mu = \{\mu_n\}$ such that μ_n belongs to $L^{\infty}(X_n)$ for each n and the norm $\|\mu\|_{\infty} = \sup\{\|\mu_n\|_{\infty} : n \geq 1\}$ is finite. Let $M(X)$ be the open unit ball of $L^{\infty}(X)$; note that if μ belongs to $M(X)$ then μ_n belongs to $M(X_n)$ for all $n \geq 1$ (but the converse statement is false). As before, the Teichmüller metric on $M(X)$ is defined by the formula:

$$\rho_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \overline{\mu}\nu} \right\|_{\infty}, \qquad \mu \text{ and } \nu \text{ in } M(X), \tag{1.5.1}$$

and it again induces on $M(X)$ the same topology that $M(X)$ inherits from $L^{\infty}(X)$.

For each $n \geq 1$ let Φ_n be the canonical projection from $M(X_n)$ to $Teich(X_n)$. For μ in $M(X)$ let $\Phi(\mu)$ be the sequence $\{\Phi_n(\mu_n)\}$. It is easy to see that $\Phi(\mu)$ belongs to $Teich(X)$ and that the map Φ from $M(X)$ to $Teich(X)$ is surjective. We call Φ the *canonical projection* of $M(X)$ onto $Teich(X)$. We have

$$d_T(s,t) = \inf\{\rho_M(\mu,\nu) : \mu \text{ and } \nu \text{ in } M(X), \Phi(\mu) = s, \text{ and } \Phi(\nu) = t\} \tag{1.5.2}$$

for all s and t in $Teich(X)$. Once again we have

$$d_T(\Phi(\mu),t) = \min\{\rho_M(\mu,\nu) : \nu \in M(X) \text{ and } \Phi(\nu) = t\}$$

for all μ in $M(X)$ and t in $Teich(X)$. Therefore, the topologies on $M(X)$ and $Teich(X)$ determined by their Teichmüller metrics make Φ an open continuous map.

The following result was proved in [8] and [15].

Theorem 1.19. *There is a unique complex Banach manifold structure on $Teich(X)$ the map $\Phi\colon M(X) \to Teich(X)$ is a holomorphic split submersion.*

Corollary 1.20. *For each $n \geq 1$ the map $t \mapsto t_n$ from $Teich(X)$ to $Teich(X_n)$ is a holomorphic split submersion.*

Proof. This is true since the projections $\Phi\colon M(X) \to Teich(X)$, $\Phi_n\colon M(X_n) \to Teich(X_n)$ and $\mu \mapsto \mu_n$ from $M(X)$ to $M(X_n)$ are holomorphic split submersions. $\square$

The tangent space at the basepoint. Let $L^1(X)$ be the complex Banach space of sequences $f = \{f_n\}$ such that f_n belongs to $L^1(X_n)$ for each n and the norm $\|f\|_1 = \sum_{n=1}^{\infty} \|f_n\|_1$ is finite. We can identify $L^\infty(X)$ with the dual space of $L^1(X)$ by sending μ in $L^\infty(X)$ to the linear functional

$$\ell_\mu(f) = \sum_{n=1}^{\infty} \iint_{X_n} \mu_n(z) f_n(z)\, dx\, dy, \qquad f \in L^1(X).$$

By definition, $A(X)$ is the closed subspace of $L^1(X)$ consisting of the φ such that φ_n belongs to $A(X_n)$ (i.e. is a holomorphic function) for each n. The orthogonal complement of $A(X)$ in $L^\infty(X)$ is by definition the set

$$A(X)^\perp = \{\mu \in L^\infty(X) : \ell_\mu(\varphi) = 0 \text{ for all } \varphi \text{ in } A(X)\}.$$

Proposition 1.21 (Generalized Teichmüller Lemma)**.** $\ker(\Phi'(0)) = A(X)^\perp$.

Proof. It is easy to verify that $\mu \in \ker(\Phi'(0))$ if and only if $\mu_n \in \ker(\Phi'_n(0))$ for each n and also that $\mu \in A(X)^\perp$ if and only if $\mu_n \in A(X_n)^\perp$ for each n. Therefore the proposition follows immediately from Proposition 1.14. □

Corollary 1.22. *The tangent space to* $Teich(X)$ *at its basepoint is naturally isomorphic to the dual space* $A(X)^*$ *of* $A(X)$*.*

The natural isomorphism sends the tangent vector $v = \Phi'(0)\mu$ to the linear functional $\varphi \mapsto \ell_\mu(\varphi)$ in $A(X)^*$.

1.6 Teichmüller space of a closed set in the sphere

In this section we define the Teichmüller space $T(E)$ of a closed subset E in $\hat{\mathbb{C}}$ and discuss some of its properties. Remember our blanket assumption that 0,1, and ∞

belong to E.

Definition 1.23. The normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$. The *basepoint* of $T(E)$ is the E-equivalence class of the identity map.

As in §1.1, for each $\mu \in M(\mathbb{C})$ we denote by w^μ the unique normalized quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself that has Beltrami coefficient μ. The *quotient map* P_E of $M(\mathbb{C})$ onto $T(E)$ is defined by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ. This is surjective and basepoint preserving. (We will use the same notation 0 for the basepoint in $M(\mathbb{C})$ and the basepoint in $T(E)$.)

The product space $Teich(E^c) \times M(E)$. We will use the symbol E^c for the complement $\widehat{\mathbb{C}} \setminus E$ of E in $\widehat{\mathbb{C}}$. If E^c is not empty, it is the union of its connected components X_n, each of which has a Teichmüller space $Teich(X_n)$. If the number of components is finite, $T(E^c)$ is by definition the cartesian product of the spaces $Teich(X_n)$. If there are infinitely many components, then E^c is the disjoint union of the countably many X_n, and $Teich(E^c)$ is the product Teichmüller space defined in §1.5. In all these cases $Teich(E^c)$ is a complex Banach manifold. Recall that $M(E)$ is the open unit ball in $L^\infty(E)$; the product space $Teich(E^c) \times M(E)$ is a complex Banach manifold.

Definition 1.24. If E^c is not empty the projection map $\mathcal{P}_E$ from $M(\mathbb{C})$ to

$Teich(E^c) \times M(E)$ is defined by the formula

$$\mathcal{P}_E(\mu) = (\Phi(\mu_{E^c}), \mu_E) \qquad \text{for all } \mu \text{ in } M(\mathbb{C}). \tag{1.6.1}$$

In the above formula, μ_{E^c} and μ_E are the restrictions of μ to E^c and E respectively and $\Phi\colon M(E^c) \to Teich(E^c)$ is the canonical projection defined in §1.5. If E^c is empty, $\mathcal{P}_E$ is the identity map of $M(\mathbb{C})$ onto itself.

Proposition 1.25 (Lieb). *For all μ and ν in $M(\mathbb{C})$ we have $P_E(\mu) = P_E(\nu)$ if and only if $\mathcal{P}_E(\mu) = \mathcal{P}_E(\nu)$. Thus, there is a well defined bijection $\theta : T(E) \to T(E^c) \times M(E)$ such that $\theta \circ P_E = \mathcal{P}_E$, and $T(E)$ has a unique complex manifold structure such that P_E is a holomorphic split submersion and the map θ is biholomorphic.*

See [8] or [15] for a complete proof. This means that there is a canonical biholomorphism between $T(E)$ and $Teich(E^c) \times M(E)$.

The space $T(E)$ is contractible.

Proposition 1.26. *There is a basepoint preserving continuous map s from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

See Proposition 7.22 in [8]

Corollary 1.27. *The space $T(E)$ is contractible.*

Proof. Since $M(\mathbb{C})$ is contractible, it follows that $T(E)$ is contractible. □

The tangent space at the basepoint. Let $A(E)$ be the closed subspace of $L^1(\mathbb{C})$ consisting of the functions f in $L^1(\mathbb{C})$ whose restriction to E^c is holomorphic.

We identify $L^\infty(\mathbb{C})$ with the dual space of $L^1(\mathbb{C})$ in the usual way. Set

$$A(E)^\perp = \{\mu \in L^\infty(\mathbb{C}) : \ell_\mu(f) = \iint_{\mathbb{C}} \mu(z) f(z)\, dx\, dy = 0 \text{ for all } f \text{ in } A(E)\}.$$

Proposition 1.28 (Teichmüller's lemma for $T(E)$)**.** $\ker(P'_E(0)) = A(E)^\perp$.

See Proposition 7.18 in [8].

Corollary 1.29. *The tangent space to $T(E)$ at its basepoint is naturally isomorphic to $A(E)^*$.*

The natural isomorphism sends the tangent vector $P'_E(0)\mu$ to the linear functional $f \mapsto \ell_\mu(f)$ on $A(E)$.

Changing the basepoint. Let h be a normalized quasiconformal self-mapping of $\widehat{\mathbb{C}}$, and let $\widetilde{E} = h(E)$. By definition, the *allowable map* h^* from $T(\widetilde{E})$ to $T(E)$ maps the $\widetilde{E}$-equivalence class of g to the E-equivalence class of $g \circ h$ for every normalized quasiconformal self-mapping g of $\widehat{\mathbb{C}}$.

Proposition 1.30. *The allowable map $h^* : T(\widetilde{E}) \to T(E)$ is biholomorphic. If μ is the Beltrami coefficient of h, then h^* maps the basepoint of $T(\widetilde{E})$ to the point $P_E(\mu)$ in $T(E)$.*

Proof. As in Proposition 1.17, let $\widetilde{h}$ be the map that sends μ in $M(\mathbb{C})$ to the Beltrami coefficient of $w^\mu \circ h$. Again $P_{\widetilde{E}}$ and P_E are holomorphic split submersions, and the standard computation shows that $\widetilde{h}$ is a biholomorphic map. □

Forgetful maps. If E is a subset of the closed set $\widetilde{E}$ and μ is in $M(\mathbb{C})$, then the $\widetilde{E}$-equivalence class of w^μ is contained in the E-equivalence class of w^μ. Therefore, there is a well-defined *forgetful map* $p_{\widetilde{E},E}$ from $T(\widetilde{E})$ to $T(E)$ such that $P_E = p_{\widetilde{E},E} \circ P_{\widetilde{E}}$.

Proposition 1.31. *The forgetful map $p_{\widetilde{E},E}$ is a basepoint preserving holomorphic split submersion.*

Proof. Since $P_E = p_{\widetilde{E},E} \circ P_{\widetilde{E}}$ and P_E and $P_{\widetilde{E}}$ are holomorphic split submersions, so is $p_{\widetilde{E},E}$. □

The following proposition will be very crucial in our thesis.

Proposition 1.32. *Let f be any holomorphic map of Δ into $T(E)$ and let μ be any point in $M(\mathbb{C})$ such that $P_E(\mu) = f(0)$. There is a holomorphic map $\widehat{f}$ from Δ to $M(\mathbb{C})$ such that $\widehat{f}(0) = \mu$ and $P_E \circ \widehat{f} = f$.*

See Proposition 7.27 in [8]. For a different approach we also refer to the papers [14] and [3].

The Kobayashi and Teichmüller metrics on $T(E)$. By definition, the Teichmüller metric $d_{T(E)}$ on $T(E)$ is given by

$$d_{T(E)}(P_E(\mu), t) = \inf\{\rho_M(\mu, \nu) : \nu \in M(\mathbb{C}) \qquad \text{and } P_E(\nu) = t\}$$

for all μ in $M(\mathbb{C})$ and t in $T(E)$.

The infinitesimal Teichmüller metric $F_{T(E)}$ is defined on the tangent bundle of $T(E)$ by the formula

$$F_{T(E)}(P_E(\mu), v) = \inf\{K_M(\mu, \lambda) : \lambda \in L^\infty(\mathbb{C}) \qquad \text{and } P'_E(\mu)\lambda = v\},$$

for any μ in $M(\mathbb{C})$ and tangent vector v to $T(E)$ at the point $P_E(\mu)$.

Proposition 1.33. *The Teichmüller and Kobayashi metrics on $T(E)$ are equal, and the infinitesimal Teichmüller and Kobayashi metrics are also equal.*

See Proposition 7.30 in [8].

The following proposition is obvious.

Proposition 1.34. *If E is a subset of $\widetilde{E}$ and $p_{\widetilde{E},E}\colon T(\widetilde{E}) \to T(E)$ is the forgetful map then*

$$d_{T(E)}(p_{\widetilde{E},E}(s), p_{\widetilde{E},E}(t)) \leq d_{T(\widetilde{E})}(s,t)$$

for all s and t in $T(\widetilde{E})$.

We defined a natural isomorphism mapping the tangent space to $T(E)$ at its basepoint at its basepoint onto a Banach space $A(E)^*$. That isomorphism is an isometry with respect to the infinitesimal Teichmüller metric on the tangent space and the usual norm on $A(E)^*$. Throughout this thesis we will denote this infinitesimal Teichmüller norm by ℓ_μ; so ℓ_μ is the norm of the linear functional

$$\ell_\mu(\phi) = \iint_{\mathbb{C}} \mu\phi dxdy \qquad \text{on } A(E).$$

Henceforth, we will denote this by

$$\|\ell_\mu\|_{T(E)} = \sup_{\|\phi\|=1} \left\{ \left| \iint_{\mathbb{C}} \mu\phi dxdy \right|, \phi \in A(E) \right\}.$$

It is clear that $\|\ell_\mu\|_{T(E)} \leq \|\mu\|_\infty$ for μ in $L^\infty(\mathbb{C})$. We say that μ is *infinitesimally extremal* in its E-equivalence class, if $\|\ell_\mu\|_{T(E)} = \|\mu\|_\infty$.

A Beltrami coefficient μ in $M(\mathbb{C})$ is called *extremal* in its E-equivalence class, if $P_E(\mu) = P_E(\nu)$ and $\|\mu\|_\infty \leq \|\nu\|_\infty$. Equivalently, μ in $M(\mathbb{C})$ is extremal in its E-equivalence class if $d_{T(E)}(0_T, P_E(\mu)) = \rho_M(0, \mu)$.

When E is finite. Let E be a finite set (as usual, 0, 1, and ∞ belong to E). Its complement $E^c = \Omega$ is the Riemann sphere with punctures at the points of E. We already saw after Proposition 1.25 that there exists a biholomorphism between $T(E)$ and $Teich(E^c) \times M(E)$. When E is finite, it follows that $T(E)$ is canonically identified with $Teich(\widehat{\mathbb{C}} \setminus E)$. For the reader's convenience we give an independent proof given in Example 3.1 in [17].

Proposition 1.35. *Let E be a finite subset of $\widehat{\mathbb{C}}$ with $\{0, 1, \infty\} \subset E$. Its compliment $\Omega = \widehat{\mathbb{C}} \setminus E$ is a sphere with finitely many punctures and there is a natural identification of $T(E)$ with the classical Teichmüller space $Teich(\Omega)$.*

Proof. We define a map $\theta : T(E) \to Teich(\Omega)$ by setting $\theta(P_E(\mu))$ equal to the Teichmüller class of the restriction of w^μ to Ω. Suppose the restrictions of w^μ and w^ν to Ω are in the same Teichmüller class. Then there exists a conformal map h of $w^\mu(\Omega)$ onto $w^\nu(\Omega)$ such that $(w^\nu)^{-1} \circ h \circ w^\mu$ is isotopic to the identity rel E. The map h is the identity since it fixes $0, 1$ and ∞. Therefore $P_E(\mu) = P_E(\nu)$ and hence θ is injective.

The map from $M(\mathbb{C})$ to $M(\Omega)$ defined as $\mu \mapsto \mu|_\Omega$ is bijective and $\theta(P_E(\mu)) = \Phi(\mu|_\Omega)$ for all μ in $M(\mathbb{C})$. Hence θ is surjective. This formula also shows θ is biholomorphic, since P_E and Φ induce the complex structures of $T(E)$ and $Teich(\Omega)$.

□

Under this indentification $d_{T(E)}$ becomes the (classical) Teichmüller metric for $Teich(\Omega)$. Furthermore, the norm of ℓ_μ is simply the norm of the linear functional that μ induces on the Banach space of integrable holomorphic functions on Ω.

Approximations by finite subsets. Let E be infinite and let $E_1, E_2, \cdots, E_n, \cdots$ be a sequence of finite subsets of E such that $\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ and $\cup_{n=1}^{\infty} E_n$ is dense in E. Let 0 be the basepoint of $T(E)$, and for each $n \geq 1$, let π_n be the forgetful map p_{E,E_n} from $T(E)$ to $T(E_n)$. For any τ in $T(E)$ and $n \geq 1$ let $\tau_n = \pi_n(\tau)$. In particular, $0_n = \pi_n(0)$ is the basepoint of $T(E_n)$ for all $n \geq 1$. By Proposition 1.34, we have

$$d_{T(E_n)}(0_n, \tau_n) \leq d_{T(E_{n+1})}(0_{n+1}, \tau_{n+1}) \leq d_{T(E)}(0, \tau)$$

for all τ in $T(E)$ and $n \geq 1$.

The following two facts will be important in our thesis. For proofs, see [17] and [18].

Proposition 1.36. *For each τ in $T(E)$ the increasing sequence $\{d_{T(E_n)}(0_n, \tau_n)\}$*

converges to $d_{T(E)}(0, \tau)$.

Proposition 1.37. *Let the infinite closed set* E *and the finite subsets* E_n, $n \geq 1$, *be as above, and let* μ *belong to* $L^\infty(\mathbb{C})$. *The sequence* $\{\|\ell_\mu\|_{T(E_n)}\}$ *is increasing and converges to* $\|\ell_\mu\|_{T(E)}$.

Chapter 2

Teichmüller contraction on $T(E)$

The following form of Teichmüller contraction was proved in [6] for classical Teichmüller spaces, and we generalise the result here for $T(E)$

Theorem A (Teichmüller contraction in $T(E)$)**.** *Let $\mu \in M(\mathbb{C})$, $P_E(\mu) = \tau \in T(E)$ and μ_0 be extremal in the E-equivalence class of μ; that is $P_E(\mu_0) = P_E(\mu)$ and $\|\mu_0\|_\infty \leq \|\mu\|_\infty$. Let $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$, also*

$$K = \frac{1+k}{1-k} \qquad \textit{and} \qquad K_0 = \frac{1+k_0}{1-k_0},$$

We define ℓ_μ as in §1.6

$$\ell_\mu(\phi) = \iint_{\mathbb{C}} \mu\phi dxdy \quad, \forall \phi \in A(E), \quad \forall \mu \in L^\infty(\mathbb{C}).$$

and

$$\|\ell_\mu\|_{T(E)} = \sup\{|\ell_\mu(\phi)|, \phi \in A(E), \|\phi\|_1 = 1\}.$$

Then

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq K - K_0.$$

Proof of Theorem A. We will assume E is infinite because if not, then $T(E)$ is isomorphic to $Teich(\widehat{\mathbb{C}} \setminus E)$ and we have the result due to Earle, proved in [6] for classical Teichmüller spaces.

So let us assume E is infinite and also $\{0, 1, \infty\} \subset E$. As in §1.6 let us consider the following sequence of closed sets in $\widehat{\mathbb{C}}$,

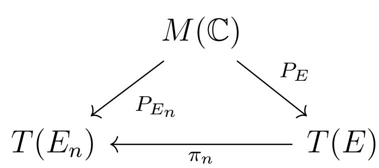
$$\{0, 1, \infty\} \subset E_1 \subset E_2 \subset E_3.... \subset E_n \subset E_{n+1}.... \subset E,$$

where each E_n is finite and $\bigcup_n E_n$ is dense in E.

We know that for all E_n we have a Teichmüller space $T(E_n)$, and let P_{E_n} denote the quotient map from $M(\mathbb{C})$ to $T(E_n)$.

Since $E_n \subset E, \forall n$ we know that if two quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$, f and g are E-equivalent they must be E_n-equivalent as well, hence there are more points in $T(E)$ than in $T(E_n)$.

So as in §1.6 we have a holomorphic, basepoint preserving, forgetful map $\pi_n : T(E) \to T(E_n)$.



That is there exists a basepoint preserving holomorphic map $\pi_n : T(E) \to (T(E_n)$ such that $P_{E_n} = \pi_n \circ P_E$.

Similarly there exist forgetful maps $\pi_{n,n+1} : T(E_{n+1}) \to T(E_n)$ such that $P_{E_n} = \pi_{n,n+1} \circ P_{E_{n+1}}$ and $\pi_n = \pi_{n,n+1} \circ \pi_{n+1}$.

Let $P_E(0) = 0$ be the basepoint of $T(E)$, $P_E(\mu) = \tau$ and $\tau_n = \pi_n(\tau)$. So we get, $0_n = \pi_n(0)$ is the basepoint of $T(E_n)$.

From §1.6 we know forgetful maps are weakly distance decreasing, and

$$d_{T(E_n)}(0_n, \tau_n) \leq d_{T(E_{n+1})}(0_{n+1}, \tau_{n+1}) \leq d_{T(E)}(0, \tau),$$

$\forall \tau \in T(E)$.

Also from Proposition 1.36 and 1.37 $\{d_{T(E_n)}(0_n, \tau_n)\}_n$ is an increasing sequence with

$$\lim_{n\to\infty} d_{T(E_n)}(0_n, \tau_n) = d_{T(E)}(0, \tau),$$

and $\forall \mu \in L^\infty(\mathbb{C})$, $\{\|\ell_\mu\|_{T(E_n)}\}_n$ is an increasing sequence with

$$\lim_{n\to\infty} \|\ell_\mu\|_{T(E_n)} = \|\ell_\mu\|_{T(E)}.$$

Given all these let us consider $\tau \in T(E)$. We know that $\tau_n = \pi_n(\tau)$, let $\tau_n = P_{E_n}(\mu_0(n))$ and $\mu_0(n)$ be extremal in its E_n-*equivalence* class. Let $k_0(n) = \|\mu_0(n)\|_\infty$ and

$$K_0(n) = \frac{1 + k_0(n)}{1 - k_0(n)}.$$

Since $T(E_n)$ is identified with $Teich(\widehat{\mathbb{C}} \setminus E_n)$ by Earle's form of Teichmüller contraction for classical Teichmüller spaces (see [6]) the following is true for all E_n,

$$\frac{1}{K_0(n)} - \frac{1}{K} \le \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E_n)}) \le K - K_0(n). \tag{2.0.1}$$

Now since $\mu_0(n)$ is extremal in its E_n-*equivalence* class, we have

$$d_{T(E_n)}(0_n, \tau_n) = \|\mu_0(n)\|_\infty = k_0(n).$$

Also, since μ_0 is extremal in its E-*equivalence* class

$$d_{T(E)}(0, \tau) = \|\mu_0\|_\infty = k_0.$$

Since $\{d_{T(E_n)}(0_n, \tau_n)\}_n$ is an increasing sequence and

$$\lim_{n\to\infty} d_{T(E_n)}(0_n, \tau_n) = d_{T(E)}(0, \tau),$$

we get $\{k_0(n)\}_n$ is an increasing sequence and so is $\{K_0(n)\}_n$. Hence

$$\lim_{n\to\infty} K_0(n) = K_0.$$

We already know that $\{\|\ell_\mu\|_{T(E_n)}\}_n$ is an increasing sequence and

$$\lim_{n\to\infty} \|\ell_\mu\|_{T(E_n)} = \|\ell_\mu\|_{T(E)}.$$

So taking limits on the equation 2.0.1 we get

$$\frac{1}{K_0} - \frac{1}{K} \le \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \le K - K_0. \tag{2.0.2}$$

□

Corollary 2.1 (Hamilton-Krushkaĺ-Reich-Strebel extremality condition for $T(E)$)**.**

A Beltrami coefficient μ is extremal in its E-equivalence class if and only if it is infinitesimally extremal in its E-equivalence class.

Proof. From the last theorem we know that

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq K - K_0.$$

From the inequalities, if $k_0 < k$ then $1 + k_0 < 1 + k$ and $1 - k_0 > 1 - k$ and $\frac{1}{1-k_0} < \frac{1}{1-k}$.

So

$$\frac{1+k_0}{1-k_0} < \frac{1+k}{1-k}.$$

And hence $K_0 < K$ and $\frac{1}{K_0} > \frac{1}{K}$ and $\frac{1}{K_0} - \frac{1}{K} > 0$.

So from the inequalities we conclude

$$\frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \geq \frac{1}{K_0} - \frac{1}{K} > 0.$$

Since $\frac{2}{1-k^2} > 0$ we conclude $(k - \|\ell_\mu\|_{T(E)}) > 0$ and hence

$$\|\ell_\mu\|_{T(E)} < k.$$

Now let's assume $\|\ell_\mu\|_{T(E)} < k.$; then $(k - \|\ell_\mu\|_{T(E)}) > 0$ and similarly

$$\frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) > 0.$$

From the inequalities we obtain

$$K - K_0 \geq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) > 0,$$

and that gives us $K - K_0 > 0$ and hence $K > K_0$.

So

$$\begin{aligned}
&K > K_0 \\
&\Rightarrow \frac{1+k}{1-k} > \frac{1+k_0}{1-k_0} \\
&\Rightarrow (1+k)(1-k_0) > (1+k_0)(1-k) \\
&\Rightarrow 1 - k_0 + k - kk_0 > 1 - k + k_0 - kk_0 \\
&\Rightarrow k - k_0 > k_0 - k \\
&\Rightarrow 2k > 2k_0 \\
&\Rightarrow k_0 < k.
\end{aligned}$$

So we get that

$$k_0 < k \Leftrightarrow \|\ell_\mu\|_{T(E)} < k.$$

And hence

$$k_0 = k \Leftrightarrow \|\ell_\mu\|_{T(E)} = k,$$

that is, μ is *extremal* in its *E-equivalence* class if and only if μ is *infinitesimally extremal* in its *E-equivalence* class.

□

Corollary 2.2. *We follow the same notations as in Theorem A. We start with a Beltrami coefficient $\mu \in M(\mathbb{C})$ and let $P_E(\mu) = \tau$. Let μ_0 be extremal in the E-equivalence class of μ. Let $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$. If either k_0 or $\|\ell_\mu\|_{T(E)}$ is* *less than k then both are less than k and*

$$\rho_\Delta\Big(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\Big) \leq \rho_\Delta(0,k).$$

Proof. From Corollary 2.1 it is clear that $k_0 < k$ if and only if $\|\ell_\mu\|_{T(E)} < k$. Let $k_0 < k$. Then,

$$\begin{aligned}
&\frac{1}{K_0} - \frac{1}{K} \\
&= \frac{1-k_0}{1+k_0} - \frac{1-k}{1+k} \\
&= \frac{(1-k_0)(1+k) - (1-k)(1+k_0)}{(1+k_0)(1+k)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1+k-k_0-kk_0-1-k_0+k+kk_0}{(1+k)(1+k_0)} \\
&= \frac{2(k-k_0)}{(1+k_0)(1+k)}.
\end{aligned}$$

Also,

$$\begin{aligned}
&K - K_0 \\
&= \frac{1+k}{1-k} - \frac{1+k_0}{1-k_0} \\
&= \frac{(1+k)(1-k_0) - (1+k_0)(1-k)}{(1-k_0)(1-k)} \\
&= \frac{1-k_0+k-kk_0-1+k-k_0+kk_0}{(1-k_0)(1-k)} \\
&= \frac{2(k-k_0)}{(1-k_0)(1-k)}.
\end{aligned}$$

We already have (2.0.1)

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq K - K_0.$$

From the above calculations we conclude

$$\frac{2(k-k_0)}{(1+k_0)(1+k)} \leq \frac{2}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq \frac{2(k-k_0)}{(1-k_0)(1-k)}.$$

Or

$$\frac{k-k_0}{(1+k_0)(1+k)} \leq \frac{1}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \leq \frac{k-k_0}{(1-k_0)(1-k)}. \tag{2.0.3}$$

Equation 2.0.3 is essentially combination of two inequalities. From the inequality on the left, we get

$$\begin{aligned}
&\frac{k-k_0}{(1+k_0)(1+k)} \leq \frac{1}{1-k^2}(k - \|\ell_\mu\|_{T(E)}) \\
\Leftrightarrow\ &\frac{k-k_0}{1+k_0} \leq \frac{k - \|\ell_\mu\|_{T(E)}}{1-k} \\
\Leftrightarrow\ &\frac{(k-k_0)(1-k)}{1+k_0} \leq k - \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\frac{k-k^2-k_0+kk_0}{1+k_0} \leq k - \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\|\ell_\mu\|_{T(E)} \leq k - \frac{k-k^2-k_0+kk_0}{1+k_0} \\
\Leftrightarrow\ &\|\ell_\mu\|_{T(E)} \leq \frac{k+kk_0-k+k^2+k_0-kk_0}{1+k_0} \\
\Leftrightarrow\ &\|\ell_\mu\|_{T(E)} \leq \frac{k^2+k_0}{1+k_0} \\
\Leftrightarrow\ &\|\ell_\mu\|_{T(E)} \leq \frac{k(k+\frac{k_0}{k})}{1+k\frac{k_0}{k}} \\
\Leftrightarrow\ &\frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{k+\frac{k_0}{k}}{1+k\frac{k_0}{k}}.
\end{aligned}$$

So the left hand side gives us

$$\frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{k+\frac{k_0}{k}}{1+k\frac{k_0}{k}} \tag{2.0.4}$$

Now looking at the inequality on the right we find that

$$\begin{aligned}
&\frac{1}{1-k^2}(k-\|\ell_\mu\|_{T(E)}) \leq \frac{k-k_0}{(1-k_0)(1-k)} \\
\Leftrightarrow\ &\frac{k-\|\ell_\mu\|_{T(E)}}{1+k} \leq \frac{k-k_0}{1-k_0} \\
\Leftrightarrow\ &k-\|\ell_\mu\|_{T(E)} \leq \frac{(k-k_0)(1+k)}{1-k_0} \\
\Leftrightarrow\ &k-\|\ell_\mu\|_{T(E)} \leq \frac{k+k^2-k_0-kk_0}{1-k_0}
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &k-\frac{k+k^2-k_0-kk_0}{1-k_0} \leq \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\frac{k-kk_0-k-k^2+k_0+kk_0}{1-k_0} \leq \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\frac{k_0-k^2}{1-k_0} \leq \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\frac{k(\frac{k_0}{k}-k)}{1-k\frac{k_0}{k}} \leq \|\ell_\mu\|_{T(E)} \\
\Leftrightarrow\ &\frac{\frac{k_0}{k}-k}{1-k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k}.
\end{aligned}$$

So the right hand side gives us

$$\frac{\frac{k_0}{k}-k}{1-k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k}. \tag{2.0.5}$$

So combining (2.0.4) and (2.0.5) we get

$$\frac{\frac{k_0}{k}-k}{1-k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{\frac{k_0}{k}+k}{1+k\frac{k_0}{k}}. \tag{2.0.6}$$

From our calculations clearly (2.0.1) and (2.0.6) are equivalent.

Next, we consider

$$\rho_\Delta\Big(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\Big) \leq \rho_\Delta(0, k). \tag{2.0.7}$$

Equation 2.0.7 is equivalent with

$$-k \leq \frac{\frac{k_0}{k} - \frac{\|\ell_\mu\|_{T(E)}}{k}}{1 - \frac{\|\ell_\mu\|_{T(E)}}{k}\frac{k_0}{k}} \leq k. \tag{2.0.8}$$

From the left inequality in 2.0.8 we get

$$\begin{aligned}
& -k \leq \frac{\frac{k_0}{k} - \frac{\|\ell_\mu\|_{T(E)}}{k}}{1 - \frac{\|\ell_\mu\|_{T(E)}}{k}\frac{k_0}{k}} \\
\Leftrightarrow & -k + \frac{\|\ell_\mu\|_{T(E)}}{k}k_0 \leq \frac{k_0}{k} - \frac{\|\ell_\mu\|_{T(E)}}{k} \\
\Leftrightarrow & \frac{\|\ell_\mu\|_{T(E)}}{k}(1+k_0) \leq \frac{k_0}{k} + k \\
\Leftrightarrow & \frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{\frac{k_0}{k} + k}{1 + k\frac{k_0}{k}}.
\end{aligned}$$

We get

$$\frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{\frac{k_0}{k} + k}{1 + k\frac{k_0}{k}}. \tag{2.0.9}$$

From the right inequality in 2.0.8 we get

$$\begin{aligned}
&\frac{\frac{k_0}{k} - \frac{\|\ell_\mu\|_{T(E)}}{k}}{1 - \frac{\|\ell_\mu\|_{T(E)}}{k}\frac{k_0}{k}} \leq k \\
\Leftrightarrow\ & \frac{k_0}{k} - \frac{\|\ell_\mu\|_{T(E)}}{k} \leq k - \frac{\|\ell_\mu\|_{T(E)}}{k} k_0 \\
\Leftrightarrow\ & \frac{k_0}{k} - k \leq \frac{\|\ell_\mu\|_{T(E)}}{k}(1 - k_0) \\
\Leftrightarrow\ & \frac{\frac{k_0}{k} - k}{1 - k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k}
\end{aligned}$$

We obtain

$$\frac{\frac{k_0}{k} - k}{1 - k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k} \tag{2.0.10}$$

Combining (2.0.9) and (2.0.10) we get

$$\frac{\frac{k_0}{k} - k}{1 - k\frac{k_0}{k}} \leq \frac{\|\ell_\mu\|_{T(E)}}{k} \leq \frac{\frac{k_0}{k} + k}{1 + k\frac{k_0}{k}} \tag{2.0.11}$$

Clearly (2.0.6) and (2.0.11) are the same and that proves the equivalence of (2.0.1) and (2.0.7). So in conclusion, $k_0 < k$ if and only if $\|\ell_\mu\|_{T(E)} < k$ and then

$$\rho_\Delta\Big(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\Big) \leq \rho_\Delta(0, k).$$

□

Chapter 3

Holomorphic isometries from Δ to $T(E)$

Definition 3.1. A map $f : \Delta \to T(E)$ is called a holomorphic isometry if f is holomorphic and for any pair $t, t' \in \Delta$, $d_{T(E)}(f(t), f(t')) = \rho_\Delta(t, t')$.

The following theorem characterizes the holomorphic isometries, essentially it gives a sufficient condition for a holomorphic map to be an isometry. This was proved for classical Teichmüller spaces in [7]; here we extend the result to $T(E)$.

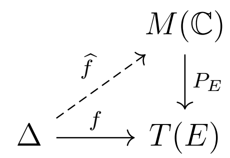
Theorem B (Holomorphic isometries from Δ to $T(E)$)**.** *Let $f : \Delta \to T(E)$ be holomorphic, and let $t_1 \in \Delta$, if*

1. $d_{T(E)}(f(t_1), f(t_2)) = \rho_\Delta(t_1, t_2)$ *for some* $t_2 \in \Delta \setminus \{t_1\}$ *or*

2. $\|f'(t_1)\|_{T(E)} = F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1-|t_1|^2}$

then f is a holomorphic isometry.

Proof of Theorem B. First consider the holomorphic lift of f as in Proposition 1.32,



By §1.6 without loss of generality one can assume $t_1 = 0$ and $f(0) = 0$. For $0 \in T(E)$ we consider the E-equivalence class of the *identity* map, that is $P_E(0)$.

Let us consider $f : \Delta \to T(E)$ where $f(0) = 0$, by the lifting in Proposition 1.32 there is a holomorphic function $\widehat{f} : \Delta \to M(\mathbb{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$.

Let us assume there is $t_2 \in \Delta \setminus \{0\}$ such that

$$d_{T(E)}(0, f(t_2)) = \rho_\Delta(0, t_2).$$

And hence we get

$$\rho_\Delta(0, t_2) = d_{T(E)}(0, f(t_2)) \leq \rho_M(0, \widehat{f}(t_2)) \leq \rho_\Delta(0, t_2)$$

where $d_{T(E)}(0, f(t_2)) \leq \rho_M(0, \widehat{f}(t_2))$ because $P_E : M(\mathbb{C}) \to T(E)$ is holomorphic, $P_E(0) = 0$ and $P_E(\widehat{f}(t_2)) = f(t_2)$ in addition $\rho_M(0, \widehat{f}(t_2)) \leq \rho_\Delta(0, t_2)$ because $\widehat{f} : \Delta \to M(\mathbb{C})$ is holomorphic and $\widehat{f}(0) = 0$.

Thus from the inequality above we get

$$d_{T(E)}(0, f(t_2)) = \rho_M(0, \widehat{f}(t_2)) = \rho_\Delta(0, t_2).$$

Hence $\widehat{f}(t_2)$ is extremal and $\|\widehat{f}(t_2)\|_\infty = |t_2|$.

Let $P : L^\infty(\mathbb{C}) \to A(E)^*$ be the linear map that takes $\mu \in L^\infty(\mathbb{C})$ to the functional ℓ_μ; as defined in §1.6, that is $P(\mu) = \ell_\mu$, and ℓ_μ is defined by

$$\ell_\mu(\phi) = \iint_{\mathbb{C}} \mu\phi dxdy,$$

where $\mu \in L^\infty(\mathbb{C})$ and $\phi \in A(E)$ and $A(E)$ is the closed subspace of $L^1(\mathbb{C})$ consisting of functions holomorphic outside E. We also know the Teichmüller norm $\|\ell_\mu\|_{T(E)} = \|\ell_\mu\|$.

Let $g : \Delta \to A(E)^*$ be defined as $g = P \circ \widehat{f}$; then $\|g(t)\| \leq \|\widehat{f}(t)\|_\infty < 1$, $\forall t \in \Delta$. Since $\forall \mu \in L^\infty(\mathbb{C})$, we have $\|\ell_\mu\| \leq \|\mu\|_\infty$. We also have $g(0) = 0$ since $\widehat{f}(0) = 0$ and $\ell_0 = 0$.

So we can apply Schwarz's Lemma for both g and $\widehat{f}$, and since $\widehat{f}(t_2)$ is extremal, it will be infinitesimally extremal by Corollary 2.1 and hence we will have

$$\|g(t_2)\| = \|\ell_{\widehat{f}(t_2)}\| = \|\widehat{f}(t_2)\|_\infty = |t_2|.$$

This is the case of equality in Schwarz's lemma, and hence we get

$$\|g^{'}(0)\| = \|\widehat{f}^{'}(0)\|_\infty = 1.$$

From the definition of P we see that $P(\mu) = 0$ if and only if $P_E^{'}(0)\mu = 0$. Using chain rule we then get

$$\|\ell_\mu\| = \inf\{\|\nu\|_\infty | \ell_\mu = \ell_\nu\} = \inf\{\|\nu\|_\infty : {P_E}^{'}(0)\mu = {P_E}^{'}(0)\nu\}$$

Hence we get

$$\|\ell_{\widehat{f}'(0)}\| = \inf\{\|\nu\|_\infty | {P_E}^{'}(0)\nu = {P_E}^{'}(0)\widehat{f}^{'}(0)\} = \inf\{\|\nu\|_\infty | {P_E}^{'}(0)\nu = f^{'}(0)\},$$

and that in turn gives

$$1 = \|\widehat{f}'(0)\|_\infty = \|\ell_{\widehat{f}'(0)}\| = F_{T(E)}(0, f'(0)).$$

Since we assumed $t_1 = 0$ and $f(0) = 0$, we obtain

$$F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1 - |t_1|^2}.$$

So

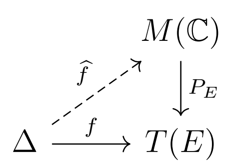
$$1 \Rightarrow 2$$

Now let us assume 2, that is there is a $t_1 \in \Delta$ such that

$$F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1 - |t_1|^2}.$$

Again without loss of generality we assume $t_1 = 0$ and $f(0) = 0$.

With our assumption we thus have $f : \Delta \to T(E)$ is a holomorphic map, and $f(0) = 0$ and $F_{T(E)}(0, f'(0)) = 1$.



Consider the lift $\widehat{f}$ again, that is a holomorphic map $\widehat{f} : \Delta \to M(\mathbb{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$. Using Schwarz's lemma as before we observe that

$$1 = F_{T(E)}(0, f'(0)) \leq \|\widehat{f}'(0)\|_\infty \leq 1$$

Clearly

$$\|\widehat{f}'(0)\|_\infty = 1.$$

Sticking with the notations before, let $P : L^\infty(\mathbb{C}) \to A(E)^*$ be the linear map that takes $\mu \in L^\infty(\mathbb{C})$ to the functional ℓ_μ, that is $P(\mu) = \ell_\mu$, where ℓ_μ is defined the usual way.

Again let $g = P \circ \widehat{f}$ that is

$$g(t) = \ell_{\widehat{f}(t)}.$$

We get

$$\|g'(0)\| = \|\widehat{f}'(0)\|_\infty = 1,$$

but this is the case of equality in Schwarz's lemma, and hence we get

$$\|g(t)\| = \|\widehat{f}(t)\|_\infty = |t|, \qquad \text{for all } t \in \Delta.$$

So for all t in Δ, $\widehat{f}(t)$ is extremal and

$$\|\widehat{f}(t)\|_\infty = |t|.$$

We see that for all t in Δ the following is true because of extremality and the last equation,

$$d_{T(E)}(0, f(t)) = d_{T(E)}(P_E(0), P_E(\widehat{f}(t))) = \rho_M(0, \widehat{f}(t)) = \rho_\Delta(0, t), \forall t \in \Delta.$$

Since $t_1 = 0$ and $f(0) = 0$ we get

$$d_{T(E)}(f(0), f(t)) = \rho_\Delta(0, t).$$

Or in other words

$$d_{T(E)}(f(t_1), f(t)) = \rho_\Delta(t_1, t), \forall t \in \Delta.$$

So $2 \Rightarrow 1$ trivially and actually does imply something stronger.

Now we need to prove $\forall t, t^{'} \in \Delta$,

$$d_{T(E)}(f(t), f(t^{'})) = \rho_\Delta(t, t^{'}).$$

If $t_1 = t^{'}$ we have nothing to prove, so let us assume $t_1 \neq t^{'}$, but we have already seen any $t \in \Delta$ could have been chosen as our t_1 and hence we can simply assume $t = t_1$ and we thus get

$$\rho_\Delta(t, t^{'}) = \rho_\Delta(t_1, t^{'}) = d_{T(E)}(f(t_1), f(t^{'})) = d_{T(E)}(f(t), f(t^{'}))$$

that is, $\forall t, t^{'} \in \Delta$,

$$d_{T(E)}(f(t), f(t^{'})) = \rho_\Delta(t, t^{'}).$$

So this proves f is a holomorphic isometry. □

Corollary 3.2. *Let $f : \Delta \to T(E)$ be a holomorphic map with $f(0) = P_E(0)$. Let $t \in \Delta \setminus \{0\}$. Define $k_0(t) = \|\nu\|_\infty$ where $f(t) = P_E(\nu)$ and ν is extremal in its E-equivalence class. We also know $\|f^{'}(0)\|_{T(E)} = F_{T(E)}(0, f^{'}(0))$. Then $k_0(t) = |t|$ if and only if $\|f^{'}(0)\|_{T(E)} = 1$.*

Proof. Let $k_0(t) = |t|$, again we look at the diagram below for reference

$$\begin{array}{ccc} & & M(\mathbb{C}) \\ & \overset{\widehat{f}}{\nearrow} & \downarrow P_E \\ \Delta & \xrightarrow{f} & T(E) \end{array}$$

We consider the holomorphic lift $\widehat{f}$ of f, that is a holomorphic map $\widehat{f} : \Delta \to M(\mathbb{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$. Then we find that

$$k_0(t) = \inf\{\|\nu\|_\infty | \nu \in M(\mathbb{C}), P_E(\nu) = f(t) = P_E \circ \widehat{f}(t)\}.$$

So from Theorem B we conclude

$$\rho_\Delta(0, t) = k_0(t) = d_{T(E)}(0, f(t)).$$

Hence f is a holomorphic isometry and

$$\|f'(0)\|_{T(E)} = 1.$$

On the other hand $\|f'(0)\|_{T(E)} = 1$ is another way of saying f is a holomorphic isometry and

$$k_0(t) = d_{T(E)}(0, f(t)) = \rho_\Delta(0, t) = |t|.$$

So $k_0(t) = |t|$ if and only if $\|f'(0)\|_{T(E)} = 1$.

□

Corollary 3.3. *Let $f : \Delta \to T(E)$ be a holomorphic map with $f(0) = P_E(0)$. Let $t \in \Delta - \{0\}$. Define $k_0(t) = \|\nu\|_\infty$ where $f(t) = P_E(\nu)$ and ν is extremal in its E-equivalence class. We also know $\|f'(0)\|_{T(E)} = F_{T(E)}(0, f'(0))$. Then if either of* *the inequalities*

1. $k_0(t) \leq |t|$

2. $\|f(t)\|_{T(E)} \leq 1$,

is strict, both are strict.

Proof. This is just a restatement of Corollary 3.2. □

Chapter 4

Schwarz's lemma for $T(E)$

We follow the same notations as before, that is $P_E(\mu) = \tau \in T(E)$, $A(E)$ is the closed subspace of $L^1(\mathbb{C})$ consisting of maps holomorphic outside E . Let $P : L^\infty(\mathbb{C}) \to A(E)^*$ be the linear map that takes $\mu \in L^\infty(\mathbb{C})$ to the functional ℓ_μ defined as

$$P(\mu)(\phi) = \ell_\mu(\phi) = \iint_{\mathbb{C}} \mu\phi dxdy, \forall \phi \in A(E)$$

with $\|\ell_\mu\|_{T(E)} = \|\ell_\mu\|$.

For a holomorphic map $f : \Delta \to T(E)$ let $k_0(t) = \|\nu\|_\infty$where $f(t) = P_E(\nu)$ and $\|\nu\|_\infty \leq \|\mu\|_\infty$, whenever $P_E(\nu) = P_E(\mu)$, that is, ν is extremal in its *E-equivalence* class.

Theorem C (Schwarz's lemma for $T(E)$)**.** *Let $f : \Delta \to T(E)$ be holomorphic and $t \in \Delta \setminus \{0\}$ with $f(0) = 0$. If either of the inequalities*

1. $\|f'(0)\|_{T(E)} \leq 1$

2. $k_0(t) \leq |t|$

is strict, then both are strict and

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \|f^{'}(0)\|_{T(E)}\Big) \leq 2\rho_\Delta(0,t).$$

Proof of Theorem C. Recall the holomorphic lift of f,

$$\begin{array}{ccc} & & M(\mathbb{C}) \\ & \overset{\widehat{f}}{\nearrow} & \downarrow P_E \\ \Delta & \xrightarrow{f} & T(E) \end{array}$$

For $f : \Delta \to T(E)$, holomorphic, basepoint preserving, we have its holomorphic lift $\widehat{f} : \Delta \to M(\mathbb{C})$, that is we have a holomorphic $\widehat{f} : \Delta \to M(\mathbb{C})$ such that $\widehat{f}(0) = 0$ and $f = P_E \circ \widehat{f}$.

Let V_0 be the Banach space of all tangent vectors at the basepoint of $T(E)$. We also know that ${P_E}^{'}(0)$ takes the tangent vectors ν in the tangent space at the basepoint of $M(\mathbb{C})$ (which is $L^\infty(\mathbb{C})$) to the functional ℓ_ν. so ${P_E}^{'}(0) \equiv P$.

So let $g = P \circ \widehat{f}$ such that $g(t) = \ell_{\widehat{f}(t)}$, then we see that $g : \Delta \to V_0$ is holomorphic and

$$f^{'}(0) = (P_E \circ \widehat{f})^{'}(0) = P_E^{'}(0)(\widehat{f}^{'}(0)) = P(\widehat{f}^{'}(0)) = \ell_{\widehat{f}^{'}(0)} = g^{'}(0),$$

as P is linear.

Let $t \in \Delta \setminus \{0\}$ be fixed and one of the following inequalities $\|f^{'}(0)\|_{T(E)} \leq 1$ and $k_0(t) \leq |t|$ be strict, then both are strict by Corollary 3.3.

So we get

$$\|g^{'}(0)\| = \|f^{'}(0)\|_{T(E)} < 1$$

and hence by Theorem 1.12 we get $\|g(t)\| < |t|$ and hence $\|\ell_{\widehat{f}(t)}\| < |t|$ or $\|\ell_{\widehat{f}(t)}\|_{T(E)} < |t|$.

By the same theorem we also get

$$\rho_\Delta\Big(\frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}, \|f^{'}(0)\|_{T(E)}\Big) \leq \rho_\Delta(0,t). \tag{4.0.1}$$

If $\|\ell_{\widehat{f}(t)}\|_{T(E)} = \|\widehat{f}(t)\|_\infty$ then by Corollary 2.1 $\widehat{f}(t)$ is extremal and $k_0(t) = \|\ell_{\widehat{f}(t)}\|_{T(E)}$ so from (4.0.1) we get

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \|f^{'}(0)\|_{T(E)}\Big) \leq \rho_\Delta(0,t). \tag{4.0.2}$$

So, suppose $\|\ell_{\widehat{f}(t)}\|_{T(E)} < \|\widehat{f}(t)\|_\infty$.

Let $r = \frac{\|\widehat{f}(t)\|_\infty}{|t|}$.

By Corollary 2.2 we know that, if $\mu_0 \in M(\mathbb{C})$ is extremal in the *E-equivalence* class of $\mu \in M(\mathbb{C})$, [that is if $P_E(\mu_0) = P_E(\mu)$ then $\|\mu_0\|_\infty \leq \|\mu\|_\infty$]. If we set $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$, then

$$\rho_\Delta\Big(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\Big) \leq \rho_\Delta(0,k).$$

So for $\mu = \widehat{f}(t)$, $k = r|t|$ and $k_0(t) = k_0$ we have

$$\rho_\Delta\Big(\frac{k_0(t)}{r|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|}\Big) \leq \rho_\Delta(0,r|t|). \tag{4.0.3}$$

Let us consider the map $\alpha : \Delta \to \Delta$ defined as $\alpha(z) = rz$, then α is holomorphic and $\alpha(0) = 0$.

Let $\frac{k_0(t)}{r|t|} = a$ and $\frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|} = b$, then $a, b \in \Delta$ and by Schwarz's lemma we get

$$\rho_\Delta(\alpha(a), \alpha(b)) \leq \rho_\Delta(a, b)$$

or

$$\rho_\Delta(ar, br) \leq \rho_\Delta(a, b)$$

or

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}\Big) \leq \rho_\Delta\Big(\frac{k_0(t)}{r|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|}\Big). \tag{4.0.4}$$

We also get

$$\rho_\Delta(0, r|t|) = \rho_\Delta(\alpha(0), \alpha|t|) \leq \rho_\Delta(0, |t|) = \rho_\Delta(0, t). \tag{4.0.5}$$

Combining (4.0.3), (4.0.4) and (4.0.5), we get

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}\Big) \leq \rho_\Delta(0, t). \tag{4.0.6}$$

Now combining (4.0.1) and (4.0.6) and applying the triangle inequality we see that

$$\begin{aligned}\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \|f'(0)\|_{T(E)}\Big) &\leq \rho_\Delta\Big(\frac{k_0(t)}{|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}\Big) + \rho_\Delta\Big(\frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}, \|f'(0)\|_{T(E)}\Big) \\ &\leq \rho_\Delta(0, t) + \rho_\Delta(0, t) = 2\rho_\Delta(0, t),\end{aligned}$$

and hence we conclude

$$\rho_\Delta\Big(\frac{k_0(t)}{|t|}, \|f'(0)\|_{T(E)}\Big) \leq 2\rho_\Delta(0, t). \tag{4.0.7}$$

□

Chapter 5

Complex geodesics in $T(E)$

Definition 5.1. A Beltrami coefficient μ in $M(\mathbb{C})$ is called *uniquely extremal* if $P_E(\nu) \neq P_E(\mu)$ whenever $\nu \in M(\mathbb{C})$, $\nu \neq \mu$, and $\|\nu\|_\infty \leq \|\mu\|_\infty$. It is easy to see that every *uniquely extremal* Beltrami coefficient is *extremal* as well.

Definition 5.2. A geodesic segment in $T(E)$ is the image of an injective continuous map $\alpha : [0,1] \to T(E)$ s.t $\forall x_0, x_1, x_2 \in [0,1]$ with $x_0 < x_1 < x_2$ we have $d_{T(E)}(\alpha(x_0), \alpha(x_2)) = d_{T(E)}(\alpha(x_0), \alpha(x_1)) + d_{T(E)}(\alpha(x_1), \alpha(x_2))$. We say α joins $P_E(\nu)$ and $P_E(\mu)$ if $\alpha(0) = P_E(\nu)$ and $\alpha(1) = P_E(\mu)$ or the other way round. Every closed subset of a geodesic segment is a geodesic segment again, provided it has more than one point.

The following proof of **Theorem D** tells us about complex geodesics in $T(E)$.

Theorem D (Complex geodesics in $T(E)$)**.** *Let $\mu_0 \in M(\mathbb{C})$, $\mu_0 \neq 0$ and μ_0 be extremal in its* E-equivalence *class. Then the following four statements are equivalent:*

1. *The Beltrami coefficient μ_0 is uniquely extremal and $|\mu_0| = \|\mu_0\|_\infty$ a.e.*

2. *There exists only one geodesic segment joining* $P_E(0)$ *and* $P_E(\mu_0)$.

3. *There exists only one holomorphic isometry* $f : \Delta \to T(E)$ *such that* $f(0) = P_E(0)$ *and* $f(\|\mu_0\|_\infty) = P_E(\mu_0)$.

4. *There exists only one holomorphic map* $g : \Delta \to M(\mathbb{C})$ *such that* $g(0) = 0$ *and* $P_E(g(\|\mu_0\|_\infty)) = P_E(\mu_0)$.

Proof of Theorem D. First we show that (2) implies (3). Let f_1 and f_2 be two holomorphic isometries from Δ to $T(E)$, such that

$$f_1(0) = f_2(0) = P_E(0)$$

and

$$f_1(\|\mu_0\|_\infty) = f_2(\|\mu_0\|_\infty) = P_E(\mu_0).$$

As they are isometries, geodesics in Δ are sent to geodesics in $T(E)$, so the image of the line segment $[0, \|\mu_0\|_\infty]$ is pointwise the same under both f_1 and f_2, since by (2) there is only one geodesic segment joining 0 and $P_E(\|\mu_0\|_\infty)$. This implies that the holomorphic mapping $f_1 - f_2$ is identically 0 on the line segment $[0, \|\mu_0\|_\infty]$, so $f_1 - f_2$ is identically 0 on Δ, so $f_1 \equiv f_2$ on Δ.

Next we show that (1) implies (4).

Let μ_0 be extremal and $|\mu_0| = \|\mu_0\|_\infty$ a.e. Let $g : \Delta \to M(\mathbb{C})$ be a holomorphic map with $g(0) = 0$ and $P_E(g(\|\mu_0\|_\infty)) = P_E(\|\mu_0\|_\infty)$.

Since g is holomorphic, by Schwarz's lemma,

$$\|g(\|\mu_0\|_\infty)\|_\infty \leq \|\mu_0\|_\infty.$$

Since μ_0 is uniquely extremal we see that

$$g(\|\mu_0\|_\infty) = \mu_0$$

Next consider $\overline{M(\mathbb{C})}$ (the closure of $M(\mathbb{C})$ in $L^\infty(\mathbb{C})$). Consider a function f in $\overline{M(\mathbb{C})}$, with $|f(z)| = 1$ a.e. Let h be another function in $\overline{M(\mathbb{C})}$, such that $h(z) \neq 0$ in $\mathbb{C} \setminus Z_h$, where $Z_h = \{z \in \mathbb{C} : h(z) = 0\}$ and $m(Z_h) = 0$, where m denotes the usual Lebesgue measure. Let $E_f = \{z \in \mathbb{C} \ : |f(z)| \neq 1\}$. By our assumption $m(E_f) = 0$. Consider the function $f_t(z) = f(z) + th(z)$. Let $F_h = \{f_t, t \in \overline{\Delta}\}$. Let us assume $F_h \subset \overline{M(\mathbb{C})}$. For any $t \in \Delta$ define $H_t = \{z \in \mathbb{C}, |f_t(z)| > 1\}$. Let $f(z) = e^{i\theta(z)}$, $h(z) = |h(z)|e^{i\phi(z)}$ and $l(z) = \phi(z) - \theta(z)$. Also, $t = |t|e^{i\psi}$. Then we have $|f_t(z)| = \sqrt{1 + |t|^2(|h(z)|)^2 + 2|t|(|h(z)|)\cos(l(z) + \psi)}$. If $f_t \in \overline{M(\mathbb{C})}$ then $m(H_t) = 0$ and if $z \in \mathbb{C} \setminus H_t$ then $1 + |t|^2(|h(z)|)^2 + 2|t||h(z)|\cos(l(z) + \psi) \leq 1$. But then $|t|^2(|h(z)|)^2 + 2|t||h(z)|\cos(l(z)+\psi) \leq 0$. Which implies $\cos(l(z)+\psi) \leq -\frac{|t||h(z)|}{2}$ or $-\cos(l(z) + \psi) \geq \frac{|t||h(z)|}{2}$ or $-\cos l(z)\cos\psi - \sin l(z)\sin\psi \geq \frac{|t||h(z)|}{2}$. Consider the functions f_1, f_i, f_{-1} and f_{-i}. Consider the set $G = E_f \cup Z_h \cup H_1 \cup H_i \cup H_{-1} \cup H_{-i}$. By our assumption $m(G) = 0$ and if $z \in \mathbb{C} \setminus G$ then $h(z) \neq 0$ and

$$-\cos l(z) \geq \frac{|h(z)|}{2}, \ -\sin l(z) \geq \frac{|h(z)|}{2},$$
$$\cos l(z) \geq \frac{|h(z)|}{2}, \ \sin l(z) \geq \frac{|h(z)|}{2}.$$

This is impossible. So at least one of the following functions, f_1, f_{-1}, f_i and f_{-i} does not belong to $\overline{M(\mathbb{C})}$. This implies that the set $F_h = \{f + th : |t| \leq 1\}$ is not a subset of $\overline{M(\mathbb{C})}$. This implies f is a *complex extreme point* of $\overline{M(\mathbb{C})}$.

The strong maximal modulus principle [Proposition 6.19 in [5]] says that if, for a holomorphic function $F : \Delta \to \overline{M(\mathbb{C})}$, there is $z \in \Delta$ such that $F(z)$ is a complex extreme point, then F is a constant function. We intend to use this for our proof.

Now let us come back to the proof of (1) implies (4). Define

$$\lambda = \frac{\mu_0}{\|\mu_0\|_\infty}.$$

Since $|\mu_0| = \|\mu_0\|_\infty \ \ a.e$, we get that

$$|\lambda| = 1 \ \ a.e,$$

Hence λ is a complex extreme point for $\overline{M(\mathbb{C})}$.

Now define $h : \Delta \to M(\mathbb{C})$ as,

$$h(t) = \begin{cases} \frac{g(t)}{t} & \text{if } t \neq 0 \\ g'(0) & \text{if } t = 0. \end{cases}$$

Then h is holomorphic and

$$h(\|\mu_0\|_\infty) = \lambda.$$

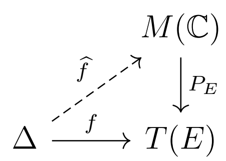
By the strong maximum modulus principle, we get $h(t) = \lambda$. This implies

$$g(t) = t\lambda = \frac{t\mu_0}{\|\mu_0\|_\infty}.$$

Since μ_0 is uniquely extremal, g is uniquely determined. We have proved (1) implies

(4).

Next we prove that (4) implies (3). Let $f : \Delta \to T(E)$ be a holomorphic isometry such that $f(0) = P_E(0)$ and $f(\|\mu_0\|_\infty) = P_E(\mu_0)$. Consider the holomorphic lift of f, that is $\widehat{f} : \Delta \to M(\mathbb{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$.



Then

$$P_E(\widehat{f}(\|\mu_0\|_\infty)) = P_E(\mu_0).$$

So the uniqueness condition in (4) says that

$$\widehat{f}(t) = \frac{t\mu_0}{\|\mu_0\|_\infty}, \quad t \in \Delta.$$

This implies

$$f(t) = P_E\Big(\frac{t\mu_0}{\|\mu_0\|_\infty}\Big), \quad t \in \Delta.$$

So f is uniquely determined and (3) is proved.

Next we show that (3) implies (1). For this, let us first prove that if (3) holds then $|\mu_0| = \|\mu_0\|_\infty$ *a.e.*

Let $r \in (0,1)$ and $Z_r = \{z \in \mathbb{C} \ : \ |\mu_0(z)| < r\|\mu_0\|_\infty\}$, we need to show that $m(Z_r) = 0$. Let χ_r be the characteristic function of Z_r. Let $\phi \in A(E)$, where $A(E)$ is the closed subspace of $L^1(\mathbb{C})$ consisting of maps holomorphic in E^c.

Define functions $f_1 : \Delta \to T(E)$ and $f_r : \Delta \to T(E)$ by

$$f_1(t) = P_E\Big(\frac{t\mu_0}{\|\mu_0\|_\infty}\Big)$$

and

$$f_r(t) = P_E\Big(\frac{t\mu_0}{\|\mu_0\|_\infty} + \frac{1-r}{2}t(t - \|\mu_0\|_\infty)\Big(\chi_r\frac{|\phi|}{\phi}\Big)\Big)$$

These maps are holomorphic and we also have

$$f_1(0) = f_r(0) = 0$$

and

$$f_1(\|\mu_0\|_\infty) = f_r(\|\mu_0\|_\infty) = P_E(\mu_0)$$

and they are isometries since

$$\rho_\Delta(0, \|\mu_0\|_\infty) = d_{T(E)}(0, P_E(\mu_0)).$$

So by (3) they coincide and we obtain

$$0 = f_1'(0) - f_r'(0) = \frac{1-r}{2}\|\mu_0\|_\infty P_E'(0)\Big(\chi_r\frac{|\phi|}{\phi}\Big).$$

This implies

$$P_E'(0)\Big(\chi_r\frac{|\phi|}{\phi}\Big) = 0.$$

Since $P_E'(0)(\mu) = \ell_\mu$, the above implies

$$\ell_{\left(\chi_r\frac{|\phi|}{\phi}\right)} = 0.$$

In particular

$$\ell_{\left(\chi_r \frac{|\phi|}{\phi}\right)}(\phi) = 0.$$

So,

$$\iint_{Z_r} |\phi| dxdy = 0.$$

This implies $m(Z_r) = 0$ since ϕ is an arbitrary non trivial function in $A(E)$.

Let $Z = \bigcup_{r \in \mathbb{Q} \cap (0,1)} Z_r$, then $m(Z) = 0$

This shows that

$$|\mu_0| = \|\mu_0\|_\infty \quad a.e.$$

For any quasiconformal homeomorphism h of $\widehat{\mathbb{C}}$, we define its Beltrami coefficient as

$$\mu_h = \frac{h_{\bar{z}}}{h_z}.$$

If h and j are two quasiconformal homeomorphisms, we have the composition formula

$$\mu_{h \circ j} = \frac{\mu_j + (\mu_h \circ j)\alpha_j}{1 + \bar{\mu_j}(\mu_h \circ j)\alpha_j}$$

where

$$\alpha_j = \frac{|j_z|^2}{(j_z)^2}.$$

If $\nu \in M(\mathbb{C})$ then by w^ν we mean the unique normalized quasiconformal homeomorphism with Beltrami coefficient

$$\mu_{w^\nu} = \nu, a.e.$$

Now let us assume (3) again. Let $\nu \in M(\mathbb{C})$ such that $\|\nu\|_\infty \leq \|\mu_0\|_\infty$ and $P_E(\nu) = P_E(\mu_0)$. Then, since μ_0 extremal, it follows that ν is also extramal and $\|\nu\|_\infty = \|\mu_0\|_\infty$. Hence $f(t) = P_E\Big(\frac{t\nu}{\|\nu\|_\infty}\Big)$ is a holomorphc isometry. So by (3) we obtain

$$P_E\Big(\frac{t\nu}{\|\nu\|_\infty}\Big) = P_E\Big(\frac{t\mu_0}{\|\mu_0\|_\infty}\Big).$$

Since ν is extremal, by (3) we obtain

$$|\nu| = \|\nu\|_\infty = \|\mu_0\|_\infty \quad a.e.$$

Also, $P_E(s\nu) = P_E(s\mu_0)$, for any s in $(0,1)$.

So $(w^{s\mu_0})^{-1} \circ w^{s\nu}$ is isotopic to the identity *rel* E.

This implies $w^{\mu_0} \circ (w^{s\mu_0})^{-1} \circ w^{s\nu}$ is isotopic to w^{μ_0} *rel* E.

This implies $(w^{\mu_0})^{-1} \circ w^{\lambda}$ is isotopic to the identity *rel* E, where

$$w^{\lambda} = w^{\mu_0} \circ (w^{s\mu_0})^{-1} \circ w^{s\nu}.$$

This implies $P_E(\lambda) = P_E(\mu_0)$.

Now let $h = w^{\mu_0} \circ (w^{s\mu_0})^{-1}$ and $j = w^{s\mu_0}$ such that $h \circ j = w^{\mu_0}$. Applying the composition formula we see that,

$$\begin{aligned} \mu_0 &= \frac{s\mu_0 + (\mu_h \circ j)\alpha_j}{1 + s\bar{\mu_0}(\mu_h \circ j)\alpha_j} \\ \Rightarrow (\mu_h \circ j)\alpha_j &= \frac{\mu_0 - s\mu_0}{1 - s|\mu_0|^2} \\ \Rightarrow |\mu_h \circ j| &= \frac{|\mu_0|(1-s)}{1 - s|\mu_0|^2}. \end{aligned}$$

We know that $|\mu_0| = \|\mu_0\|_\infty$ $a.e.$ Let $\|\mu_0\|_\infty = k$ and $sk = k^{'}$; we can rewrite the composition formula as

$$|\mu_h \circ j| = \frac{k - k^{'}}{1 - kk^{'}} = k^{''} \quad a.e.$$

So $|\mu_h| = k^{''}$ $a.e$, since j is quasiconformal and thus absolutely continuous. Also we get $k = \frac{k^{'} + k^{''}}{1 + k^{'}k^{''}}$.

Now let us consider $h = w^{\mu_0} \circ (w^{s\mu_0})^{-1}$ and $j = w^{s\nu}$ so that $h \circ j = w^{\lambda}$. By similar calculations we obtain

$$\lambda = \frac{s\nu + (\mu_h \circ j)\alpha_j}{1 + s\bar{\nu}(\mu_h \circ j)\alpha_j}.$$

Since $|s\nu| = k^{'}$ $a.e$ and $|\mu_h \circ j| = k^{''}$ $a.e$ and $|\alpha_j| = 1$, we can say $s\nu = k^{'}e^{i\theta}$ $a.e$ and $(\mu_h \circ j)\alpha_j = k^{''}e^{i\phi}$ $a.e.$ Hence,

$$\lambda = \frac{k^{'}e^{i\theta} + k^{''}e^{i\phi}}{1 + k^{'}e^{-i\theta}k^{''}e^{i\phi}}$$
$$\Rightarrow \lambda = e^{i\theta}\frac{k^{'} + k^{''}e^{i(\phi-\theta)}}{1 + k^{'}k^{''}e^{i(\phi-\theta)}}$$
$$\Rightarrow \lambda = e^{i\theta}\frac{k^{'} + k^{''}e^{il}}{1 + k^{'}k^{''}e^{il}},$$

where $l = \phi - \theta$. Hence, $|\lambda| = \left|\frac{k^{'} + k^{''}e^{il}}{1 + k^{'}k^{''}e^{il}}\right|$.

Now, observe,

$$
\begin{aligned}
&\left|\frac{k^{'}+k^{''}e^{il}}{1+k^{'}k^{''}e^{il}}\right| \leq \frac{k^{'}+k^{''}}{1+k^{'}k^{''}} \\
&\Leftrightarrow \frac{(k^{'}+k^{''}\cos l)^{2}+k^{''2}\sin^{2}l}{(1+k^{'}k^{''}\cos l)^{2}+k^{'2}k^{''2}\sin^{2}l} \leq \frac{k^{'2}+2k^{'}k^{''}+k^{''2}}{1+2k^{'}k^{''}+k^{'2}k^{''2}} \\
&\Leftrightarrow \frac{k^{'2}+2k^{'}k^{''}\cos l+k^{''2}}{1+2k^{'}k^{''}\cos l+k^{'2}k^{''2}} \leq \frac{k^{'2}+2k^{'}k^{''}+k^{''2}}{1+2k^{'}k^{''}+k^{'2}k^{''2}} \\
&\Leftrightarrow (k^{'2}+k^{''2})(1+k^{'2}k^{''2})+2k^{'}k^{''}(k^{'2}+k^{''2})+2k^{'}k^{''}(1+k^{'2}k^{''2})\cos l \\
&+4k^{'2}k^{''2}\cos l \\
&\leq (k^{'2}+k^{''2})(1+k^{'2}k^{''2})+2k^{'}k^{''}(k^{'2}+k^{''2})\cos l+2k^{'}k^{''}(1+k^{'2}k^{''2}) \\
&+4k^{'2}k^{''2}\cos l \\
&\Leftrightarrow (k^{'2}+k^{''2})+(1+k^{'2}k^{''2})\cos l \leq (k^{'2}+k^{''2})\cos l+(1+k^{'2}k^{''2}) \\
&\Leftrightarrow (k^{'2}+k^{''2})(1-\cos l)-(1+k^{'2}k^{''2})(1-\cos l) \leq 0 \\
&\Leftrightarrow (k^{'2}-1)(1-k^{''2})(1-\cos l) \leq 0 \\
&\Leftrightarrow (1-k^{'2})(1-k^{''2})(1-\cos l) \geq 0.
\end{aligned}
$$

The last inequality is true since $k^{'}<1$, $k^{''}<1$, and $\cos l \leq 1$. So we get

$$|\lambda| \leq \frac{k^{'}+k^{''}}{1+k^{'}k^{''}} = k.$$

Since we already had $P_E(\mu_0) = P_E(\lambda)$ and μ_0 is extremal, we get that λ is extremal as well. Hence $|\lambda| = k \;\; a.e.$ But this implies

$$\left|\frac{k^{'}+k^{''}e^{il}}{1+k^{'}k^{''}e^{il}}\right| = \frac{k^{'}+k^{''}}{1+k^{'}k^{''}}.$$

This implies

$$(1-k^{'2})(1-k^{''2})(1-\cos l)=0.$$

Since $k^{'}<1$ and $k^{''}<1$, this can happen if and only if $\cos l=1$, i.e $\cos(\phi-\theta)=1$. This implies $s\nu=k^{'}e^{i\theta}$ $a.e$ and $(\mu_h\circ j)\alpha_j=k^{''}e^{i\phi}$ $a.e$ have the same argument and can be rewritten as $s\nu=k^{'}e^{i\theta}$ $a.e$ and $(\mu_h\circ j)\alpha_j=k^{''}e^{i\theta}$ $a.e.$

We can write $(\mu_h\circ j)\alpha_j=m.s\nu$ where $m=\frac{k^{'}}{k^{''}}>0$, So

$$\lambda=\nu\frac{s+ms}{1+ms^2k^2}.$$

This shows that λ is a positive multiple of ν. Let us write (for simplicity) $\lambda = p\nu\ where\ p>0$. So $\|\lambda\|_\infty=p\|\nu\|_\infty$, But we already know that $\|\lambda\|_\infty=\|\nu\|_\infty=\|\mu_0\|_\infty=k>0$. So $p=1$ and hence $\lambda=\nu$ $a.e.$ So

$$w^\nu=w^\lambda=w^{\mu_0}\circ(w^{s\mu_0})^{-1}\circ w^{s\nu}\quad a.e.$$

$$\Rightarrow w^\nu\circ(w^{s\nu})^{-1}=w^{\mu_0}\circ(w^{s\mu_0})^{-1}\quad a.e$$

Since $s\in(0,1)$ was arbitrary, letting $s\to 0$, we observe $w^\nu=w^{\mu_0}$ $a.e$ and hence $\nu=\mu_0$ $a.e.$ This proves that μ_0 is uniquely extremal. So we proved (3) implies (1).

Finally we show that (1) implies (2).

Let μ_0 be uniquely extremal, and $|\mu_0|=\|\mu_0\|_\infty=k$ $a.e.$ Let $\alpha:[0,1]\to T(E)$ be an injective continuous map, defined by $\alpha(t)=P_E(t\mu)$, so that $\alpha([0,1])$ is a geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$. We intend to show this is the only geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$.

Let us assume that there is another injective continuous map $\beta : [0,1] \to T(E)$, such that $\beta([0,1])$ is another geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$.

Let $\nu \in M(\mathbb{C})$ be a point such that $P_E(\nu) \in \beta([0,1]) \setminus \alpha([0,1])$.

Let ν_0 be extremal in the E-*equivalence* class of ν.

Since $P_E(\nu_0)$ is an interior point of the geodesic segment we see that

$$d_{T(E)}(P_E(0), P_E(\nu_0)) \le d_{T(E)}(P_E(0), P_E(\mu_0). \tag{5.0.1}$$

Since $|\mu_0| = k$ *a.e*, and ν_0 is extremal, we can see $|\mu_0| \ge |\nu_0|$ *a.e.*

Consider the mapping $w^\eta = w^{\mu_0} \circ (w^{\nu_0})^{-1}$, so that $w^\eta \circ w^{\nu_0} = w^{\mu_0}$.

Let η_0 be the extremal in the E-*equivalence* class of η. Observe that $w^{\eta_0} \circ w^{\nu_0} = w^{\widetilde{\mu}}$ for some $\widetilde{\mu}$ such that $P_E(\widetilde{\mu}) = P_E(\mu_0)$. So we get

$$|\eta \circ w^{\nu_0}| = \left| \frac{\mu_0 - \nu_0}{1 - \overline{\nu_0}\mu_0} \right|$$

and

$$|\eta_0 \circ w^{\nu_0}| = \left| \frac{\widetilde{\mu} - \nu_0}{1 - \overline{\nu_0}\widetilde{\mu}} \right|.$$

Let $\|\widetilde{\mu}\|_\infty = n$, $\|\nu_0\|_\infty = l$. Since μ_0 and ν_0 are both extremal we get that $l < k \le n$.

Now consider the map

$$f(z) = \frac{z-a}{1-\bar{a}z}, \qquad a \in \Delta.$$

This map is holomorphic in Δ and $f(a) = 0$. So, if $1 > \delta_1 > \delta_2 > a$, then $a \in \overline{B_{\delta_2}(0)} \subset B_{\delta_1}(0)$, where $B_\delta(0) = \{z \in \Delta \ : \ |z-a| < \delta\}$. Since f is a Möbius transformation, by maximum modulus principle,

$$\delta_1 > \delta_2 \Leftrightarrow \sup_{z\in B_{\delta_1}(0)} |f(z)| = \sup_{z\in \partial B_{\delta_1}(0)} |f(z)| > \sup_{z\in \partial B_{\delta_2}(0)} |f(z)| = \sup_{z\in B_{\delta_2}(0)} |f(z)|$$

and

$$\delta_1 = \delta_2 \Leftrightarrow \sup_{z\in B_{\delta_1}(0)} |f(z)| = \sup_{z\in \partial B_{\delta_1}(0)} |f(z)| = \sup_{z\in \partial B_{\delta_2}(0)} |f(z)| = \sup_{z\in B_{\delta_2}(0)} |f(z)|.$$

Applying this to our problem we see that for all possible values of ν_0, since $\|\widetilde{\mu}\|_\infty = n$ and $n \geq k$, we have

$$\sup_{\widetilde{\mu}} \left| \frac{\widetilde{\mu} - \nu_0}{1 - \overline{\nu_0}\widetilde{\mu}} \right| \geq \sup_{\mu_0} \left| \frac{\mu_0 - \nu_0}{1 - \overline{\nu_0}\mu_0} \right|.$$

So

$$\sup_{\nu_0} \sup_{\widetilde{\mu}} \left| \frac{\widetilde{\mu} - \nu_0}{1 - \overline{\nu_0}\widetilde{\mu}} \right| \geq \sup_{\nu_0} \sup_{\mu_0} \left| \frac{\mu_0 - \nu_0}{1 - \overline{\nu_0}\mu_0} \right|.$$

This implies

$$\|\eta_0\|_\infty \geq \|\eta\|_\infty.$$

But, since $\|\eta_0\|_\infty \leq \|\eta\|_\infty$, we conclude

$$\|\eta_0\|_\infty = \|\eta\|_\infty.$$

From our discussion above we see that $n = k$, that is $\|\mu_0\|_\infty = \|\widetilde{\mu}\|_\infty$. We conclude $\widetilde{\mu}$ is extremal, and since μ_0 is uniquely extremal, $\widetilde{\mu} = \mu_0$. So

$$w^{\eta_0} = w^{\mu_0} \circ (w^{\nu_0})^{-1}.$$

This gives us

$$d_{T(E)}(P_E(0), P_E(\eta_0)) = d_{T(E)}(P_E(\nu_0), P_E(\mu_0)). \tag{5.0.2}$$

Since $P_E(0)$, $P_E(\nu_0)$ and $P_E(\mu_0)$ are on a geodesic segment, we have

$$d_{T(E)}(P_E(0), P_E(\nu_0)) + d_{T(E)}(P_E(\nu_0), P_E(\mu_0)) = d_{T(E)}(P_E(0), P_E(\mu_0)).$$

Using Equation (5.0.2) we get

$$d_{T(E)}(P_E(0), P_E(\nu_0)) + d_{T(E)}(P_E(0), P_E(\eta_0)) = d_{T(E)}(P_E(0), P_E(\mu_0)).$$

Since μ_0, ν_0, and η_0 are extremal in their respective classes, we get

$$\rho_\Delta(0, \|\nu_0\|_\infty) + \rho_\Delta(0, \|\eta_0\|_\infty) = \rho_\Delta(0, \|\mu_0\|_\infty). \tag{5.0.3}$$

This implies

$$\|\eta_0\|_\infty = \frac{\|\mu_0\|_\infty - \|\nu_0\|_\infty}{1 - \|\nu_0\|_\infty \|\mu_0\|_\infty}. \tag{5.0.4}$$

But since we already have

$$w^{\eta_0} = w^{\mu_0} \circ (w^{\nu_0})^{-1},$$

we get that

$$|\eta_0 \circ w^{\nu_0}| = \left|\frac{\mu_0 - \nu_0}{1 - \bar{\nu_0}\mu_0}\right|. \tag{5.0.5}$$

Let $\nu_0 = s\mu_0$, $s = |s|e^{i\phi}$ and $\mu_0 = ke^{i\theta}$ and $|s| < 1$. From Equation (5.0.4) then we get,

$$\|\eta_0\|_\infty = k\frac{1 - \sup|s|}{1 - \sup|s|k^2}. \tag{5.0.6}$$

From Equation (5.0.5) we get that

$$|\eta_0 \circ w^{\nu_0}| = k\left|\frac{1-|s|e^{i(\phi-\theta)}}{1-|s|k^2e^{i(\theta-\phi)}}\right|. \tag{5.0.7}$$

Setting $\omega = \phi - \theta$, we rewrite that as

$$|\eta_0 \circ w^{\nu_0}| = k\left|\frac{1-|s|e^{i\omega}}{1-|s|k^2e^{-i\omega}}\right|. \tag{5.0.8}$$

Now observe,

$$\begin{aligned}
&\left|\frac{1-|s|e^{i\omega}}{1-|s|k^2e^{-i\omega}}\right| \geq \frac{1-|s|}{1-|s|k^2}\\
&\Leftrightarrow \frac{(1-|s|\cos\omega)^2+|s|^2\sin^2\omega}{(1-|s|k^2\cos\omega)^2+|s|^2k^4\sin^2\omega} \geq \frac{(1-|s|)^2}{(1-|s|k^2)^2}\\
&\Leftrightarrow \frac{1-2|s|\cos\omega+|s|^2}{1-2|s|k^2\cos\omega+|s|^2k^4} \geq \frac{1-2|s|+|s|^2}{1-2|s|k^2+|s|^2k^4}\\
&\Leftrightarrow (1-2|s|\cos\omega+|s|^2)(1-2|s|k^2+|s|^2k^4)\\
&\geq (1-2|s|+|s|^2)(1-2|s|\cos\omega k^2+|s|^2k^4)\\
&\Leftrightarrow (1+|s|^2)(1+|s|^2k^4)+(1+|s|^2)(-2|s|k^2)\\
&-2|s|\cos\omega(1+|s|^2k^4)+4|s|^2k^4\cos\omega
\end{aligned}$$

$$\geq (1+|s|^2)(1+|s|^2k^4) + (1+|s|^2)(-2|s|\cos\omega k^2)$$

$$-2|s|(1+|s|^2k^4) + 4|s|^2k^4\cos\omega$$

$$\Leftrightarrow -2|s|(k^2+|s|^2k^2) - 2|s|\cos\omega(1+|s|^2k^4)$$

$$\geq -2|s|\cos\omega(k^2+|s|^2k^2) - 2|s|(1+|s|^2k^4)$$

$$\Leftrightarrow (1+|s|^2k^4)(1-\cos\omega)$$

$$\geq (k^2+|s|^2k^2)(1-\cos\omega)$$

$$\Leftrightarrow (1+|s|^2k^4-k^2-|s|^2k^2)(1-\cos\omega) \geq 0$$

$$\Leftrightarrow (1-k^2)(1-|s|^2k^2)(1-\cos\omega) \geq 0.$$

The last inequality is true since $k < 1$, $s < 1$ and $\cos\omega \leq 1$. So Equation (5.0.7) gives

$$|\eta_0 \circ w^{\nu_0}| \geq k\frac{1-|s|}{1-|s|k^2}.$$

Hence

$$\|\eta_0\|_\infty \geq k\frac{1-|s|}{1-|s|k^2}. \tag{5.0.9}$$

So from Equations (5.0.6) and (5.0.9) we obtain

$$k\frac{1-\sup|s|}{1-\sup|s|k^2} \geq k\frac{1-|s|}{1-|s|k^2}$$

$$\Rightarrow (\sup|s|-|s|)k^2 \geq (\sup|s|-|s|)$$

$$\Rightarrow (\sup|s|-|s|)(k^2-1) \geq 0$$

$$\Rightarrow \sup|s| - |s| \leq 0$$

$$\Rightarrow \sup|s| \leq |s|$$

$$\Rightarrow |s| = \sup|s| = S.$$

So from Equations (5.0.6) and (5.0.8) we get

$$k\frac{1-S}{1-Sk^2} = \|\eta_0\|_\infty \geq |\eta_0 \circ w^{\nu_0}| = k\left|\frac{1-Se^{i\omega}}{1-Sk^2e^{i\omega}}\right| \geq k\frac{1-S}{1-Sk^2}.$$

So

$$k\left|\frac{1-Se^{i\omega}}{1-Sk^2e^{i\omega}}\right| = k\frac{1-S}{1-Sk^2}.$$

But that is true if and only if

$$(1-k^2)(1-S^2k^2)(1-\cos\omega) = 0.$$

That can happen only when $\cos\omega = \cos(\phi-\theta) = 1$, which means ν_0 and μ_0 have the same argument and hence we can write $\nu_0 = S\mu_0, 1 > S > 0$. But that is a contradiction to our assumption. So we conclude that there is only one geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$. Hence we have proved (1) implies (2) and completed the proof of the theorem.

□

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